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Nowhere-zero 3-flows in toroidal graphs

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Tutte's 3-flow conjecture states that every 4-edge-connected graph admits a nowhere-zero 3-flow. The planar case of Tutte's 3-flow conjecture is the classical Grötzsch's Theorem (1959). Steinberg and Younger (1989) further verified Tutte's 3-flow conjecture for projective planar graphs. In this paper we confirm Tutte's 3-flow conjecture for all toroidal graphs.

1. Introduction

The concept of integer flow was originally introduced by Tutte [14,15] as the dual of graph coloring. Tutte in 1972 proposed the following well-known 3-*flow conjecture*.

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3-Flow Conjecture: Every 4-edge-connected graph admits a nowhere-zero 3-flow.

This conjecture was confirmed for planar graphs, which is the dual of Grötzsch's Theorem [2] in 1959 that every triangle-free planar graph is 3-vertex-colorable. Steinberg and Younger [9] in 1989 further verified it for projective planar graphs. How about toroidal graphs? This natural question was asked implicitly in the survey "The state of the three color problem" [10] by Steinberg in 1993.

The corresponding problem for toroidal graphs was solved by Thomassen [11] in 1994 as follows.

Theorem 1.1. (Thomassen [11]) *Every loopless graph embedded on torus without contractible cycles of length at most four is* 3*-vertex-colorable.*

The dual version of Theorem 1.1 for flow problems, by a result of Tutte [15] (Theorem 4.2 below), is restated as follows.

Corollary 1.1. *For a graph G embedded on torus, if its dual graph G^* is loopless and contains no contractible cycles of length at most four, then G admits a nowhere-zero* 3*-flow.*

It was pointed out by Thomassen [4] that, in order to verify the 3-Flow Conjecture for toroidal graphs, one more result is needed for *nearly-planar graphs* beyond Theorem 1.1. Here a graph G is called *nearly-planar* if there is an edge $e \in E(G)$ such that $G - e$ is planar. This problem proposed by Thomassen is proved in this paper as one of the main results.

Theorem 1.2. *Every* 4*-edge-connected nearly-planar graph admits a nowhere-zero* 3*-flow.*

Aided with Corollary 1.1 and Theorem 1.2, the 3-Flow Conjecture is verified for toroidal graphs as follows.

Theorem 1.3. *Every* 4*-edge-connected toroidal graph admits a nowhere-zero* 3*-flow.*

Note that Theorem 1.3 is not a straightforward consequence of Corollary 1.1 and Theorem 1.2. For some technical reasons, it is rather complicated to apply some lifting techniques for 4-edge-connected graphs embedded on torus, in which the embedding property may not be preserved. In fact, we will prove a more convenient and stronger version of Theorem 1.3 concerning odd edge connectivity, where the definition of odd edge connectivity can be found in Section 2.

Theorem 1.4. *Every odd-*5*-edge-connected toroidal graph admits a modulo* 3*-orientation.*

We outline the proof of this theorem as follows. By a lifting lemma for odd edge connectivity (Lemma 4.1), any minimal counterexample must be a 5-regular toroidal graph G. The proof is divided into three cases according to the structure of the dual graph G^* of G on torus. In the first case, we suppose that G^* contains neither loops nor contractible 4-cycles, then Corollary 1.1 implies that G admits a nowhere-zero 3-flow. In the second case, we suppose that G^* contains a loop, then by the above mentioned discovery of Thomassen, the dual edge of this loop in G is a handle-edge e such that $G-e$ is planar, which implies that G is a nearly-planar graph and it admits a nowhere-zero 3-flow by Theorem 1.2. In the last case, we assume instead that G^* contains a contractible 4-cycle C. In this case the dual of C is a 4-edge-cut in G and we use flow extension arguments of planar graphs to yield a smaller counterexample, completing the proof. See Section 4 for more details.

2. Preliminaries

In this section, we introduce some necessary notation and terminology. Graphs in this paper are finite, while multiple edges are allowed. For additional notation and terminology we follow [1,16]. Let $G = (V, E)$ be a graph and let k be a positive integer. We use $[k]$ to denote the set $\{1, 2, \ldots, k\}$. A vertex of degree k (at most k, respectively) in G is called a *k-vertex* (*k^--vertex*, respectively), and we denote by $V_k(G)$ ($V_{k^-}(G)$, respectively) the set of all k-vertices (k^--vertices, respectively). An *edge-cut* is a minimal set of edges whose deletion increases the number of components in a graph. Denote by *k-edge-cut* (*k^--edge-cut*, respectively) an edge-cut of size k (at most k, respectively).

For vertices u and v of G, we denote by $\mu_G(u, v)$ the number of parallel edges joining u and v. The multiplicity $\mu(G)$ of G is defined as $\mu(G) = \max\{\mu_G(u, v) : \{u, v\} \subseteq V(G)\}$. We use $E_G(v)$ to denote the set of all edges incident with v in G. The neighborhood of v and closed neighborhood of v are denoted by $N_G(v) = \{x : E_G(x) \cap E_G(v) \neq \emptyset, x \neq v\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. For vertex subsets $U, W \subseteq V(G)$, let $[U, W]_G = (\bigcup_{u \in U} E_G(u)) \cap (\bigcup_{w \in W} E_G(w))$. When $U = \{u\}$ or $W = \{w\}$, for convenience, we write $[u, W]_G$ or $[U, w]_G$ for $[U, W]_G$, respectively. The subgraph of G induced by $S \subseteq V(G)$ is denoted by $G[S]$. For any subset $S \subseteq V(G)$, we denote $S^c = V(G) - S$ and set $d_G(S) = |[S, S^c]_G|$. A graph is called *odd-k-edge-connected* if it contains no $(2t-1)$-edge-cut for any $1 \leq t \leq \lfloor \frac{k}{2} \rfloor$. An edge-cut $[S, S^c]_G$ in a connected graph G is called *essential* if both $G[S]$ and $G[S^c]$ contain nonloop edges. Moreover, a connected graph is called *essentially k-edge-connected* if it contains no essential $(k-1)^-$-edge-cut.

A cycle C of a plane graph is called *separating* if each of its interior and exterior contains at least one vertex. A separating k-cycle is a separating cycle of length k. Two edges in a plane graph G are called consecutive if they are adjacent in the boundary of a face of G.

A graph G is called *nearly-planar* if there exists an edge $e \in E(G)$ such that $G - e$ is planar. Such an edge is called a *handle-edge*. We use $\mathcal{N}$ to denote the collection of all nearly-planar graphs. Note that every planar graph is nearly-planar, and every edge can be viewed as a handle-edge in a planar graph. A graph is called *toroidal* if it can be embedded on the torus. A cycle in an embedded graph is called *contractible* if one of its faces is homeomorphic to a disc.

Let $D = D(G)$ be an orientation of G. For each $v \in V(G)$, we use $E_D^+(v)$ and $E_D^-(v)$ to denote the set of all arcs directed out of v and directed into v, respectively. Denote $d_D^+(v) = |E_D^+(v)|$ and $d_D^-(v) = |E_D^-(v)|$. If $d_D^+(v) - d_D^-(v) \equiv 0 \pmod 3$ for each $v \in V(G)$, then D is called a *modulo* 3*-orientation* of G. For an Abelian group A, an ordered pair (D, f) is called an A*-flow* of G if D is an orientation and f is an edge-mapping from $E(G)$ to A such that every vertex $v \in V(G)$ is balanced, that is

$$\sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e) = 0.$$

An A-flow (D, f) is *nowhere-zero* if $f(e) \in A \setminus \{0\}$, $\forall e \in E(G)$. Clearly, modulo 3-orientations and nowhere-zero $\mathbb{Z}_3$-flows are equivalent as $2 = -1$ in $\mathbb{Z}_3$. A nowhere-zero $\mathbb{Z}$-flow is called a *nowhere-zero* k*-flow* if $0 < |f(e)| < k$ for each $e \in E(G)$. A theorem of Tutte [15] shows that for all graphs, the existence of a nowhere-zero $\mathbb{Z}_k$-flow is equivalent to the existence of a nowhere-zero k-flow. Therefore in what follows, we shall study nowhere-zero 3-flows in terms of modulo 3-orientations. The advantage of modulo 3-orientations is allowing to consider the orientations of graphs only, and ignore the edge-mapping.

3. Nearly-planar graphs and flow extensions

3.1. Flow extensions

A graph G is called $\mathcal{M}_3$*-extendable* at a vertex $u \in V(G)$ if any pre-orientation D_u at $E_G(u)$ with $d_{D_u}^+(u) \equiv d_{D_u}^-(u) \pmod 3$ can be extended to a modulo 3-orientation D of the entire graph G. Inspired by the works of Steinberg and Younger [9], Thomassen [13] and Lovász et al. [8], and using refined discharging arguments, we shall prove the following key lemma as part of the proof of Theorem 1.4.

Lemma 3.1. *If G is a 5-edge-connected nearly-planar graph, then G is $\mathcal{M}_3$-extendable at any 7^--vertex incident with a handle-edge.*

Note that Lemma 3.1 also holds for planar graphs since every planar graph is a nearly-planar graph by definition. The planar case of Lemma 3.1 will be used in the proof of Theorem 1.4 in Section 4.

For induction purposes, we shall apply Lemma 3.1 to prove the following lemma, which is slightly stronger than Theorem 1.2 and allows 2-edge-cuts.

Lemma 3.2. *Every odd-5-edge-connected nearly-planar graph admits a modulo 3-orientation.*

3.2. Some facts and lemmas for nearly-planar graphs

Next we introduce two operations, contracting and lifting, as our main tools for the proof. To *lift* a pair of edges vv_1 and vv_2 incident to a vertex v in a graph G means to delete vv_1 and vv_2, and adding a new edge $e' = v_1v_2$ joining v_1 and v_2. To *contract* an edge e in G means to identify two endpoints of e and then delete the resulting loops. For an edge $e \in E(G)$, we use G/e to denote the graph obtained from G by contracting e. For a subgraph H of G, we write G/H to denote the graph obtained from G by successively contracting all edges in $E(H)$.

Many basic properties of graphs are preserved after contracting and lifting operations by their definitions. We summarize these useful facts as follows, these will be frequently used implicitly in later proofs.

Observation 3.3. (**for lifting**) *Let G be a nearly-planar graph with a handle-edge e_0. Let $vv_1, vv_2 \in E(G)$ and let G' be the graph obtained from G by lifting vv_1 and vv_2. Each of the following holds.*
(i) If $e_0 \in \{vv_1, vv_2\}$, then $G' \in \mathcal{N}$ and the new edge v_1v_2 is a handle-edge of G'.
(ii) If vv_1, vv_2 are two consecutive edges in a planar embedding of $G - e_0$, then $G' \in \mathcal{N}$ and e_0 is a handle-edge of G'.
(iii) If G' admits a modulo 3-orientation, then G admits a modulo 3-orientation as well.

Observation 3.4. (**for contracting**) *Let G be a nearly-planar graph with a handle-edge e_0.*
(i) For any edge $e \in E(G)$ with $e \neq e_0$, $G/e \in \mathcal{N}$.
(ii) Let H be a subgraph of G such that $H - e_0$ is connected. Then we have the following.
(ii-a) If $e_0 \in G[V(H)]$, then G/H is a planar graph and so $G/H \in \mathcal{N}$.
(ii-b) If $e_0 \notin G[V(H)]$, then $G/H \in \mathcal{N}$ and e_0 is still a handle-edge in G/H.
(iii) For any 2-edge-connected subgraph H of G, we have $G/H \in \mathcal{N}$.

Observation 3.5. *Let G be a graph with $|V(G)| \geq 3$ and k be a positive integer. Let $e \in E(G)$. If G has no k-edge-cut, then G/e contains no k-edge-cut as well. In particular, if G is k-edge-connected, then so is G/e.*

Observation 3.6. *If G is k-edge-connected, then for any $S \subsetneq V(G)$ with $d_G(S) \leq 2k-1$, both $G[S]$ and $G[S^c]$ are connected.*

Lemma 3.7. *Every 4-edge-connected essentially 6-edge-connected plane graph G with $|V(G)| \geq 6$ contains no separating 3-cycle $uvwu$ with* $\max\{d_G(u), d_G(v), d_G(w)\} \leq 5$.

Proof. To the contrary, suppose that G has a separating 3-cycle $C = uvwu$ with $\max\{d_G(u), d_G(v), d_G(w)\} \leq 5$. Let S_1 (S_2, respectively) be the subset of $V(G)$ consisting of the vertices in the interior (exterior, respectively) of C. Note that $S_1 \cap V(C) = \emptyset$ and $S_2 \cap V(C) = \emptyset$. Clearly, $d_G(S_1) + d_G(S_2) = d_G(V(C)) \leq 9$. Since $|V(G)| \geq 6$, WLOG, we assume $|S_2| \geq 2$. Then it is clear that $d_G(S_1) \geq 4$ and $d_G(S_2) \geq 6$ since G is 4-edge-connected essentially 6-edge-connected, a contradiction. $\square$

Definition 3.8. Let G be a graph.
(i) For a mapping $\beta : V(G) \mapsto \mathbb{Z}_3$ with $\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod 3$, an orientation D of G is called a *β-orientation* if $d_D^+(v) - d_D^-(v) \equiv \beta(v) \pmod 3$ for every vertex $v \in V(G)$.
(ii) G is called *$\mathbb{Z}_3$-connected* if it admits a β-orientation for any possible mapping $\beta : V(G) \mapsto \mathbb{Z}_3$ with $\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod 3$.

A useful method to obtain modulo 3-orientations is the following lemma.

Lemma 3.9. (Lai [5]) *Let G be a graph, and let $H \subseteq G$ be a subgraph of G.*
(i) If H is $\mathbb{Z}_3$-connected and G/H has a modulo 3-orientation, then G has a modulo 3-orientation.
(ii) If both H and G/H are $\mathbb{Z}_3$-connected, then G is also $\mathbb{Z}_3$-connected.
(iii) The graphs $2K_2$ and W_4 are $\mathbb{Z}_3$-connected, where $2K_2$ consists of two vertices and two parallel edges, and W_4 is constructed by adding a new center vertex connecting to each vertex of a 4-cycle.

4. Proof of Theorem 1.4 assuming Lemmas 3.1 and 3.2

We shall prove our main result Theorem 1.4 in this section, assuming the truth of Lemmas 3.1 and 3.2. The proofs of Lemmas 3.1 and 3.2 are much more involved, and they will be postponed to later sections.

We start with the following useful lifting lemma of Zhang [17] for odd edge connectivity. The advantage of this lemma is to preserve the embedding property of the new graph after lifting.

Lemma 4.1. (Zhang [17]) *Let $G = (V, E)$ be an odd-k-edge-connected graph (where $k \geq 3$ is an odd number). Assume that there is a vertex $x \in V(G)$ such that $d(x) \neq k$. Arbitrarily label the edges of G incident with x as $\{e_1, \ldots, e_b\}$ (where $b = d(x)$). Then there is an integer $i \in \{1, \ldots, b\}$ such that the new graph obtained from G by lifting e_i and e_{i+1} (the index is taken modulo b) away from x remains odd-k-edge-connected.*

Another key result is the following fundamental flow-coloring duality of Tutte [15] for orientable surfaces, which together with Theorem 1.1 of Thomassen [11] would handle toroidal graphs whose dual graphs on torus contain neither loops nor contractible small cycles.

Theorem 4.2. (Tutte [15], see [3,16]) *If a graph G has a face-k-colorable 2-cell embedding in some orientable surface, then it has a nowhere-zero k-flow.*

Now we are ready to prove Theorem 1.4 that every odd-5-edge-connected toroidal graph admits a modulo 3-orientation.

Proof of Theorem 1.4. By way of contradiction, we assume the result is false and study a minimal counterexample G with respect to $|V(G)| + |E(G)|$. The theorem naturally holds for graphs with two vertices, and so we have $|V(G)| \geq 3$. Clearly, G is 2-connected; otherwise we use induction on all the blocks of G to result in a modulo 3-orientation of G. Note that the toroidal property and odd edge connectivity are preserved under contraction. By Lemma 3.9(i)(iii), G contains no parallel edges. It is also clear that G contains no essential 2-edge-cut $[S, S^c]_G$. Otherwise we use induction on $G/G[S^c]$ and $G/G[S]$ to obtain modulo 3-orientations D_1 and D_2, respectively. Then either D_1 and D_2 agree along $[S, S^c]_G$ directly, or they agree after reversing all edge directions in D_2. Thus, their union provides a modulo 3-orientation of G, a contradiction. Moreover, we claim that G is 5-regular by Lemma 4.1. We embed the graph G on torus and also use G to denote the embedding for notational convenience. If G is not 5-regular, then there is a vertex x whose degree is not equal to five. So we apply Lemma 4.1 to lift two consecutive edges incident to the vertex x, yield a smaller odd-5-edge-connected graph without modulo 3-orientations, a contradiction. Hence G must be a 4-edge-connected 5-regular graph.

Let G^* be the dual graph of G on torus. The proof is divided into three cases according to the structure of G^*.

Case 1. G^* contains neither noncontractible loops nor contractible 4-cycles.

Note that there exist neither contractible loops nor contractible 3-cycles in G^* since G is 4-edge-connected. Consider the underlying simple graph $\widetilde{G^*}$ of G^*, i.e., the graph obtained from G^* by deleting all parallel edges. Then $\widetilde{G^*}$ is a loopless simple graph embedded on torus without contractible cycles of length at most 4. By Theorem 1.1, we obtain that $\widetilde{G^*}$ (and G^*) is 3-vertex-colorable, and so G is face-3-colorable. If G contains a face not homeomorphic to an open disk, then G is a planar graph, and so G has a nowhere-zero 3-flow by Grötzsch's Theorem, a contradiction. Hence, we suppose that G is a 2-cell embedding on torus. As G is face-3-colorable and by Theorem 4.2, the graph G also has a nowhere-zero 3-flow, a contradiction again. This verifies Case 1, and we conclude that G^* has a noncontractible loop or a contractible 4-cycle.

Case 2. G^* contains a noncontractible loop, say e'_0.

Let e_0 be the dual edge of e'_0 in G. Observe that, if we cut the graph G^* and the torus up along e'_0, then we obtain a planar embedding. This implies that $G - e_0$ has a planar embedding and so G is nearly-planar. By Lemma 3.2, G has a modulo 3-orientation, a contradiction.

Case 3. G^* contains a contractible 4-cycle.

By the definition of contractible cycle, we obtain that G has a 4-edge-cut $[S, S^c]_G$, which is the dual of the contractible 4-cycle of G^*, such that $G/G[S^c]$ is planar and

$G/G[S]$ is toroidal. Since G is 4-edge-connected, we have that $G/G[S^c]$ is 4-edge-connected, and so $d_G(W) \geq 4$ for any $W \subseteq S$. Among all subset $W \subseteq S$ with $d_G(W) = 4$, we choose one, say T, with minimal cardinality (possibly $T = S$). That is, for any proper subset $T_1 \subsetneq T$, we have $d_G(T_1) \geq 5$ by the minimality of T. Clearly, $|T| \geq 2$ since G is 5-regular. Moreover, $G[T]$ and $G[T^c]$ are both connected, since G is 4-edge-connected. Then we have that $G/G[T^c]$ is planar and $G/G[T]$ is toroidal.

Note that $G/G[T^c]$ and $G/G[T]$ are both 4-edge-connected by Observation 3.5. Since G is a minimal counterexample, we get a modulo 3-orientation D_1 of $G/G[T]$. Then, in G, we contract $G[T^c]$ to become a new vertex u, and we assign a pre-orientation D_u which is the restriction of D_1 on those edges incident to u. Clearly, the degree of u in the planar graph $G/G[T^c]$ is 4. Then we replace one arbitrary oriented edge incident with u by two oriented edges in the opposite direction to obtain a new planar graph G' with a new pre-orientation D'_u. By the fact that $d_G(T_1) \geq 5$ for any proper subset $T_1 \subsetneq T$ from the minimality of T, the new planar graph G' is 5-edge-connected. By Lemma 3.1, G' has an $\mathcal{M}_3$-extension of D'_u. Thus, $G/G[T^c]$ has an $\mathcal{M}_3$-extension of D_u. This, together with D_1, results in a modulo 3-orientation on G, a contradiction. This completes the proof. □

5. Proof of Lemma 3.1

Our proof of Lemma 3.1 employs some flow extension ideas from Steinberg and Younger [9], Thomassen [11–13], and Lovász et al. [8] in their proofs of Grötzsch's theorem and the weak 3-flow conjecture. Moreover, some new ideas come from our work [7] on 3-flows of some signed planar graphs, which is used to deal with some negative edges in various locations. Here we realize that the handle-edge has behavior similar to a pair of negative loops, and similar but modified reductions and discharging arguments can be applied for nearly-planar graphs.

Proof outlines: First, we apply reductions to show that any minimal counterexample G is almost 5-regular and contains no small nontrivial edge-cut. Then, we use some discharging arguments to find Grötzsch Configurations in several different situations. Finally, we perform a reduction using a Grötzsch Configuration to get a smaller 5-edge-connected nearly-planar graph, leading to a contradiction and completing the proof.

We copy Lemma 3.1 here for convenience.

Lemma 3.1. If G is a 5-edge-connected nearly-planar graph, then G is $\mathcal{M}_3$-extendable at any 7^--vertex incident with a handle-edge.

Now we proceed to prove Lemma 3.1 in the following subsections.

5.1. Some prior reductions

By way of contradiction, we choose a counterexample G of Lemma 3.1 with a 7^--vertex u, a pre-orientation D_u, and a handle-edge $e_0 = uu_0$ such that

(1) $|V(G)| + |E(G - u)|$ is minimum;
(2) subject to (1), $d_G(u)$ is as large as possible;
(3) subject to (1) and (2), if G is planar, then we choose a handle-edge $e_0 = uu_0$ of G with $\mu_G(u, u_0)$ minimized among all possible choices.

Claim I. *We have $d_G(u) = 7$ and $|V(G)| \geq 4$.*

Proof. First, we claim that $d_G(u) = 7$. For otherwise, u is a 5-vertex or 6-vertex. If u is a 6-vertex, then we replace an oriented edge incident with u other than e_0 by two oriented edges in the opposite direction; if u is a 5-vertex, then we replace an oriented edge incident with u other than e_0 by two oriented edges in the same direction and one oriented edge in the opposite direction. In this way, we get another counterexample G' with $|V(G')| + |E(G' - u)|$ still minimized, which is a contradiction to the choice of (2) with $d_G(u)$ being maximized.

In addition, it is easy to check that Lemma 3.1 holds naturally when $|V(G)| = 2$ or $|V(G)| = 3$, and so we have $|V(G)| \geq 4$. □

Now, we choose a planar embedding of $G - e_0$ without separating 2-cycles and still denote this planar embedding by $G - e_0$ for notational convenience. Notice that, to avoid separating 2-cycles, it is enough to embed all parallel edges between two vertices x and y consecutively in a cyclic order for each vertex pair. In the following, the graph $G - e_0$ mentioned in context always takes the planar embedding.

Claim II. *Each of the following holds.*
(i) $G - u$ contains no $\mathbb{Z}_3$-connected subgraph of order at least 2. In particular, $G - u$ contains neither $2K_2$ nor W_4 as a subgraph by Lemma 3.9(iii).
(ii) G is 2-connected.

Proof. (i) Suppose to the contrary that $G - u$ contains a $\mathbb{Z}_3$-connected subgraph G_1 of order at least 2. Since every $\mathbb{Z}_3$-connected graph is 2-edge-connected, by Observations 3.4(ii-b) and 3.5, G/G_1 is a 5-edge-connected nearly-planar graph with the handle-edge e_0 incident with u. The minimality of G and Lemma 3.9(i) imply that G/G_1 has an $\mathcal{M}_3$-extension of D_u, and so does G, a contradiction.

(ii) Suppose to the contrary that G contains a cut-vertex v. Then we can obtain two connected nearly-planar subgraphs of G, denoted by G_1 and G_2, satisfying that $V(G_1) \cap V(G_2) = \{v\}$ and $V(G_1) \cup V(G_2) = V(G)$. By Observation 3.6, $u \neq v$. We may assume $u \in V(G_1)$. So G_1 is nearly-planar and G_2 is planar. Moreover, G_2 is simple by (i). Note that any edge-cut of G is an edge-cut of G_1 or G_2, and vice versa. So we get that G_1 and G_2 are both 5-edge-connected. The minimality of G implies that Lemma 3.1 is applicable for both G_1 and G_2. Hence G_1 has an $\mathcal{M}_3$-extension of D_u. Since G_2 is a 5-edge-connected simple planar graph, it has a modulo 3-orientation by assigning a pre-orientation to a 5-vertex of G_2 and applying Lemma 3.1. Moreover, the union of their orientations is an $\mathcal{M}_3$-extension of D_u on G, a contradiction. Hence G is 2-connected. □

Claim III. *G contains no essential 7^--edge-cut.*

Proof. By contradiction, we suppose that $[S, S^c]_G$ is an essential 7^--edge-cut. Clearly, $G[S]$ and $G[S^c]$ are both 2-edge-connected since G is 5-edge-connected and by Observation 3.6. Moreover, $G/G[S]$ and $G/G[S^c]$ are both 5-edge-connected nearly-planar graphs by Observations 3.4(iii) and 3.5.

We may assume $u \in S$. Since G is a minimal counterexample, we get an $\mathcal{M}_3$-extension D_1 of D_u on $G/G[S^c]$. Then, in G, we contract $G[S]$ to become a vertex u', and we assign a pre-orientation $D'_{u'}$ which is the restriction of D_1 on $G/G[S]$. By Observation 3.4(ii), if $u_0 \in S$, then $G/G[S]$ is a planar graph; if $u_0 \in S^c$, then $G/G[S]$ is a nearly-planar graph with the handle-edge e_0 incident with u'. For both cases above, we obtain that $G/G[S]$ has an $\mathcal{M}_3$-extension of $D'_{u'}$ by the minimality of G. This, together with D_1, results in an $\mathcal{M}_3$-extension of D_u on G, a contradiction. □

Claim IV. *Each of the following holds.*
(i) $\mu(G) \le 2$, all parallel edges belong to $E_G(u)$, and no edge is parallel to e_0.
(ii) Every vertex of G other than u is a 5-vertex.

Proof. First, we claim that $d_G(v) \le 6$ for each $v \in V(G) - \{u\}$. Suppose to the contrary that there exists a vertex $v \in V(G) - \{u\}$ with $d_G(v) \ge 7$. By Claim II(ii), the vertex v has at least two distinct neighbors. Recall that $G - e_0$ is a plane graph. If $|N_{G-e_0}(v)| \ge 2$, then we obtain a new graph G' from G by lifting a pair of edges vv_1, vv_2 incident with v in $G - e_0$ that are successive but not parallel; if $|N_{G-e_0}(v)| = 1$, then e_0 is incident with v and a new graph G' is obtained from G by lifting a pair of edges vv_1, $e_0 = vv_2$ incident with v in G that are not parallel. For both cases above, we have that $G' \in \mathcal{N}$ and there is a handle-edge incident with u in G' by Observation 3.3. Then we obtain a new pre-orientation D'_u on $E_{G'}(u)$ by restricting D_u on $E_{G'}(u)$ and setting $D'_u(v_jv_i) = D(vv_i)$ if vv_i is pre-oriented in G for $\{i, j\} = \{1, 2\}$. It is clear that $\delta(G') \ge 5$ and G' contains neither loops nor essential 5^--edge-cuts by Claim III. So G' is 5-edge-connected. Since G is a minimal counterexample, we get an $\mathcal{M}_3$-extension D' of D'_u on G', and then it provides an $\mathcal{M}_3$-extension of D_u on G by Observation 3.3(iii), a contradiction. This proves that $d_G(v) \le 6$ for each $v \in V(G) - \{u\}$.

By Claim II(i), the only possible parallel edges are in $E_G(u)$. If there exists a vertex $u_1 \in N_G(u)$ such that $\mu_G(u, u_1) \ge 3$, then for $S = \{u, u_1\}$, $d_G(S) \le 7$ and $[S, S^c]_G$ is an essential edge-cut by Claim I and Observation 3.6, a contradiction to Claim III. Hence, we have $\mu(G) \leq 2$. Clearly, if G is not planar, then there exists no edge parallel to e_0 by the definition of a handle-edge in a nearly-planar graph; if G is planar, then every edge incident with u is a handle-edge of G and so there exists no edge parallel to e_0 by the choice (3) of e_0. This proves (i).

In order to prove (ii), it suffices to show that $d_G(v) \neq 6$ for each $v \in V(G)-\{u\}$. Now, suppose that there exists a vertex $v \in V(G)-\{u\}$ with $d_G(v)=6$. By Claim IV(i) and Claim II(i), v is incident with at most two parallel edges. Similar to the above discussion, we lift all the edges incident with v in pairs successively in a way that keeps near-planarity and does not create loops. The obtained new nearly-planar graph is denoted by G' with a new pre-orientation D'_u on $E_{G'}(u)$. Clearly, $\delta(G') \geq 5$ and G' is connected since $G-v$ is connected by Claim II(ii). If there is an essential 4^--edge-cut $[S,S^c]_{G'}$ of G', then by the fact that $|[v,S]_G| \leq 3$ or $|[v,S^c]_G| \leq 3$, we have $d_G(S) \leq 7$ or $d_G(S \cup \{v\}) \leq 7$. This is a contradiction to Claim III. So G' is 5-edge-connected. Hence the minimality of G implies that G' has an $\mathcal{M}_3$-extension D' of D'_u, and thus G has an $\mathcal{M}_3$-extension of D_u by Observation 3.3(iii), a contradiction again. This verifies (ii). $\square$

In the rest of the proof, we will find a Grötzsch Configuration in $G-e_0$ to perform reduction.

5.2. Finding Grötzsch configurations

Denote by R the graph obtained from W_5 by deleting an edge in the 5-cycle.

Definition 5.1. Let G be a graph with an edge e_0 for which $G-e_0$ is a plane graph in which some of the edges are pre-oriented. Let H be a subgraph of $G-e_0$ isomorphic to R. Then H is called a *Grötzsch Configuration* of $G-e_0$ if each of the following holds:
(i) $d_G(v)=5$ for each $v \in V(H)$;
(ii) every triangle in H is a boundary of a 3-face of $G-e_0$;
(iii) for any $ab \in E(H)$, $\mu_{G-e_0}(a,b)=1$ and ab is not pre-oriented in $G-e_0$;

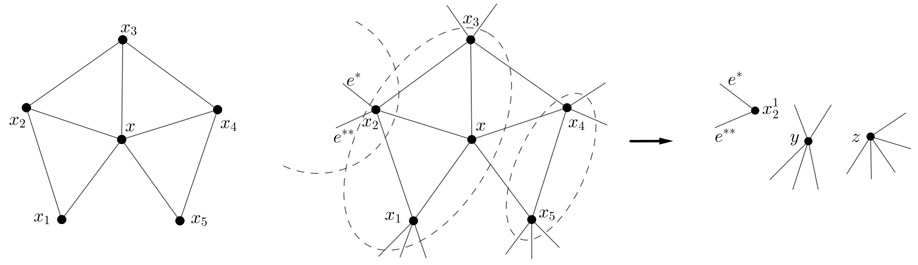
We label the vertices of H as shown in Fig. 1(a). The vertex x is called a *center* of H and vertices x_2, x_4 are called two *corners* of H. Furthermore, a vertex $v \in \{x_2,x_4\}$ is called a *good corner* of H if it is not incident with parallel edges in $G-e_0$. Denoted by $(H;v)$ the Grötzsch Configuration H with a good corner v.

Claim V. *$G-e_0$ contains a Grötzsch Configuration H with a good corner.*

Proof. Recall that $e_0=uu_0$. We start with Euler's Formula:

$$|V(G-e_0)|-|E(G-e_0)|+|F(G-e_0)|=2,$$

and the Degree Sum Formula:



(a) A Grötzsch Configuration H in $G - e_0$ (b) $(H; x_2)$ in G (c) The Grötzsch Reduction G^*

Fig. 1. A Grötzsch Configuration and some related graphs.

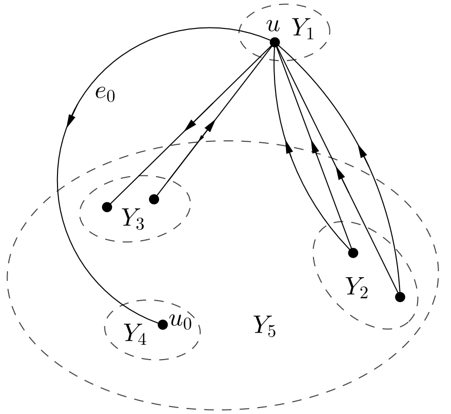


Fig. 2. A partition $\{Y_1, Y_2, Y_3, Y_4, Y_5\}$ of $V(G)$.

$$2|E(G - e_0)| = \sum_{v \in V(G-e_0)} d_{G-e_0}(v).$$

Following Lebesgue [6], we define the Euler contribution $w : V(G - e_0) \mapsto \mathbb{Q}$ with

$$w(v) = -\frac{d_{G-e_0}(v) - 2}{2} + \sum_{f \in F_{G-e_0}(v)} \frac{1}{d(f)},$$

where $F_{G-e_0}(v)$ is the set of faces incident with v and $d(f)$ is the number of edges incident with the face f in $G - e_0$. Combining Euler's Formula with the Degree Sum Formula, we have

$$\sum_{v \in V(G-e_0)} w(v) = |F(G - e_0)| - |E(G - e_0)| + |V(G - e_0)| = 2. \tag{1}$$

The vertex set $V(G - e_0) = V(G)$ can be partitioned into five subsets, denoted by $\{Y_1, Y_2, Y_3, Y_4, Y_5\}$, as shown in Fig. 2, where $Y_1 = \{u\}$, $Y_2 \subseteq N_{G-e_0}(u)$ and consists of vertices incident with precisely one digon, $Y_3 = N_{G-e_0}(u) - Y_2$, $Y_4 = \{u_0\}$ (where $e_0 =$

uu_0), and $Y_5 = V(G-e_0)-Y_1-Y_2-Y_3-Y_4$ (that is, $Y_5 = V(G-e_0)-\{u,u_0\}-N_{G-e_0}(u)$). By Definition 5.1 and Claim IV, every center of a Grötzsch Configuration of $G-e_0$ is in Y_5 and no vertex in $Y_3 \cup Y_4 \cup Y_5$ is incident with a digon.

In the remaining proof, by calculating Euler contribution of each vertex, we show that Y_5 contains at least $5-|Y_2|$ centers of Grötzsch Configurations.

Note that $|Y_1| = |Y_4| = 1$, and by Claim I and Claim IV(ii) we have

$$d_{G-e_0}(u) = 6, d_{G-e_0}(u_0) = 4, \text{ and } d_{G-e_0}(v) = 5, \forall v \in Y_2 \cup Y_3 \cup Y_5.$$

Let $|Y_2| = t$, where $0 \le t \le 3$. Then we have $|Y_3| = 6-2t$.

For $Y_1 = \{u\}$,

$$\begin{aligned} w(u) &= -\frac{d_{G-e_0}(u)-2}{2} + \sum_{f \in F_{G-e_0}(u)} \frac{1}{d(f)} \\ &\le -\frac{6-2}{2} + \frac{t}{2} + \frac{6-t}{3} = \frac{t}{6}. \end{aligned}$$

For each $v \in Y_2$,

$$\begin{aligned} w(v) &= -\frac{d_{G-e_0}(v)-2}{2} + \sum_{f \in F_{G-e_0}(v)} \frac{1}{d(f)} \\ &\le -\frac{5-2}{2} + \frac{1}{2} + \frac{4}{3} = \frac{1}{3}. \end{aligned}$$

For each $v \in Y_3$,

$$\begin{aligned} w(v) &= -\frac{d_{G-e_0}(v)-2}{2} + \sum_{f \in F_{G-e_0}(v)} \frac{1}{d(f)} \\ &\le -\frac{5-2}{2} + \frac{5}{3} = \frac{1}{6}. \end{aligned}$$

For $Y_4 = \{u_0\}$, the degree sequence of faces incident with u_0 in $G-e_0$ cannot be $(3,3,3,3)$; for otherwise, the vertex subset $\{u_0\} \cup N_{G-e_0}(u_0)$ induces a W_4, which contradicts Claim II(i). Thus, at least one of these faces is of degree at least 4. So we have

$$\begin{aligned} w(u_0) &= -\frac{d_{G-e_0}(u_0)-2}{2} + \sum_{f \in F_{G-e_0}(u_0)} \frac{1}{d(f)} \\ &\le -\frac{4-2}{2} + \frac{1}{4} + \frac{3}{3} = \frac{1}{4}. \end{aligned}$$

Hence the total Euler contribution of $Y_1 \cup Y_2 \cup Y_3 \cup Y_4$ is at most

$$\frac{t}{6} + \frac{1}{3}t + \frac{1}{6}(6-2t) + \frac{1}{4} = \frac{5}{4} + \frac{t}{6}.$$

By Equation (1), the total Euler contribution of Y_5 is at least

$$2-(\frac{5}{4}+\frac{t}{6})=\frac{3}{4}-\frac{t}{6}. \quad (2)$$

For Y_5, assume that a vertex v of Y_5 is not the center of any Grötzsch Configuration. Then there are at most three 3-faces incident with v. Hence,

$$w(v)=-\frac{d_{G-e_0}(v)-2}{2}+\sum_{f\in F_{G-e_0}(v)}\frac{1}{d(f)}\leq -\frac{5-2}{2}+\frac{2}{4}+\frac{3}{3}=0. \quad (3)$$

For a vertex $v\in Y_5$ as the center of a Grötzsch Configuration,

$$w(v)=-\frac{d_{G-e_0}(v)-2}{2}+\sum_{f\in F_{G-e_0}(v)}\frac{1}{d(f)}\leq -\frac{5-2}{2}+\frac{5}{3}=\frac{1}{6}. \quad (4)$$

Hence, by Equations (2), (3) and (4), there are at least

$$\lceil(\frac{3}{4}-\frac{t}{6})/\frac{1}{6}\rceil=\lceil\frac{9}{2}-t\rceil=5-t$$

Grötzsch Configurations with center in Y_5. By the definition of good corner, there are at most $\lfloor\frac{t}{2}\rfloor$ Grötzsch Configurations without any good corner. Since $0\leq t\leq 3$, we have $5-t>\lfloor\frac{t}{2}\rfloor$. Therefore, there exists a Grötzsch Configuration with a good corner. □

By Claim V, we may assume that x_2 is a good corner of H and $(H;x_2)$ is a Grötzsch Configuration of $G-e_0$. We assign labels e^* and e^{**} to the two edges in $E_G(x_2)-E_H(x_2)$ (see Fig. 1(b)).

Definition 5.2. Let G be a graph with an edge e_0 for which $G-e_0$ is a plane graph with $(H;x_2)$ as a Grötzsch Configuration. A *Grötzsch Reduction* of G with respect to $(H;x_2)$ is a graph, denoted by $G^*=GR(G;H;x_2)$, obtained as follows:
(1) Split the vertex x_2 into a 2-vertex x_2^1 and a 3-vertex x_2^2 such that the edges incident with x_2^1 in the new graph G_0 are e^* and e^{**}.
(2) Set $E_0=\{x_3x_4,xx_4,xx_5\}$, $Y=\{x,x_1,x_2^2,x_3\}$ and $Z=\{x_4,x_5\}$. By deleting all edges in E_0, contracting $E(G_0[Y])$ to a new vertex y, and contracting $E(G_0[Z])$ to a new vertex z, we obtain the graph $G^*=GR(G;H;x_2)$ (see Fig. 1(c)).

Remark. In Definitions 5.1 and 5.2, it might be possible that the edge e_0 in G is incident to x_i where $i\in\{1,2,3,4,5\}$. In that case, $G^*=GR(G;H;x_2)$, the Grötzsch Reduction of G with respect to $(H;x_2)$, contains the edge e_0 which is incident to one of x_2^1,y,z. It is also worth noting that, as it can be seen in the proofs of Claims VI and VII below, we always have $d_{G^*}(z)=d_{G^*}(y)=5$, since $\mu_G(x_1,x_3)=0$ by Claim III and Lemma 3.7.

Claim VI. *Let G^* be a Grötzsch Reduction of G with respect to $(H; x_2)$. Each of the following holds.*
(i) G^* is a connected nearly-planar graph with a handle-edge incident with u.
(ii) G^* contains no 4^--edge-cut other than the 2-edge-cut $[x_2^1, V(G^*) - x_2^1]_{G^*}$.

Proof. (i) By the construction of G^*, G^* is connected since G is 5-edge-connected and essentially 8-edge-connected. Moreover, by Observation 3.4(ii), we have that G^* is a nearly-planar graph with a handle-edge incident with u since $G_0[Y]$ and $G_0[Z]$ are both connected subgraphs of $G - e_0$. This proves (i).

(ii) We now show that G^* contains no 4^--edge-cut other than the 2-edge-cut $[x_2^1, V(G^*) - x_2^1]_{G^*}$. To the contrary, we suppose that G^* contains a 4^--edge-cut $[S, S^c]_{G^*}$ such that $\{x_2^1\} \notin \{S, S^c\}$. By Lemma 3.7, $G - e_0$ contains no edge joining x_i and x_j if $1 \leq i < j \leq 5$, $i + 1 \neq j$ and $i + 4 \neq j$, since G is 5-edge-connected essentially 8-edge-connected. Recall that there is no edge parallel to $e_0 = uu_0$ in G and $u \notin V(H)$. By the construction of G^*, we have $d_{G^*}(x_2^1) = 2$, $|N_{G^*}(x_2^1)| = 2$, $d_{G^*}(u) = 7$ and $d_{G^*}(v) = 5$ for each $v \in V(G^*) - \{x_2^1, u\}$. This implies that $[S, S^c]_{G^*}$ is essential. Clearly, $[S, S^c]_{G^*}$ separates the set $\{x_2^1, y, z\}$. By symmetry, we only need to consider three cases as follows.

Case 1 $x_2^1 \in S$ and $\{y, z\} \subseteq S^c$.

Note that $|S| \geq 2$. Suppose $|S| = 2$. Since x_2^1 is not incident with parallel edges and $d_{G^*}(v) \geq 5$ for each $v \in V(G^*) - \{x_2^1\}$, we have $d_{G^*}(S) \geq 5$, contrary to $d_{G^*}(S) \leq 4$. Assume instead that $|S| \geq 3$. Let $S_1 = S - \{x_2^1\}$. It is clear that $[S_1, S_1^c]_G$ is an essential edge-cut of G with $d_G(S_1) \leq d_{G^*}(S) + |\{e^*, e^{**}\}| \leq 6$ by Observation 3.6, contrary to Claim III.

Case 2 $z \in S$ and $\{x_2^1, y\} \subseteq S^c$.

Let $S_1 = (S - \{z\}) \cup \{x_4, x_5\}$. Then by Observation 3.6, we have that $[S_1, S_1^c]_G$ is an essential edge-cut of G with $d_G(S_1) = d_{G^*}(S) + |\{x_3x_4, xx_4, xx_5\}| \leq 7$, a contradiction again.

Case 3 $y \in S$ and $\{x_2^1, z\} \subseteq S^c$.

We first show that $G^*[S^c] - e_0$ contains an (x_2^1, z)-path. Suppose it does not, and assume that A_1 and B_1 are two disjoint subsets of S^c such that $G^*[A_1]$ and $G^*[B_1]$ are two components of $G^*[S^c] - e_0$ with $x_2^1 \in A_1$ and $z \in B_1$, as shown in Fig. 3. Suppose $A_1 = \{x_2^1\}$. Clearly, in G^*, there is an edge e joining x_2^1 and a vertex in S. So we have

$$d_G((B_1 - \{z\}) \cup \{x_4, x_5\}) \leq d_{G^*}(S^c) - |\{e\}| + |\{x_3x_4, xx_4, xx_5\}| + |\{e_0\}| \leq 7,$$

contrary to Claim III. Hence suppose $|A_1| \geq 2$. Then we have $d_G(A_1 - \{x_2^1\}) \geq 5$ and $d_G((B_1 - \{z\}) \cup \{x_4, x_5\}) \geq 8$ since G is 5-edge-connected and by Claim III, which contradicts $d_G(A_1 - \{x_2^1\}) + d_G((B_1 - \{z\}) \cup \{x_4, x_5\}) \leq d_G((S^c - \{x_2^1, z\}) \cup \{x_4, x_5\}) + 2|\{e_0\}| \leq d_{G^*}(S^c) + |\{e^*, e^{**}, x_3x_4, xx_4, xx_5\}| + 2|\{e_0\}| \leq 11$. Thus, $G^*[S^c] - e_0$ contains an (x_2^1, z)-path as claimed.

This implies that $G[S_1] - e_0$ contains an (x_2, x_4)-path or an (x_2, x_5)-path P, where $S_1 = (S^c - \{x_2^1, z\}) \cup \{x_2, x_4, x_5\}$. Note that $x \notin S_1$.

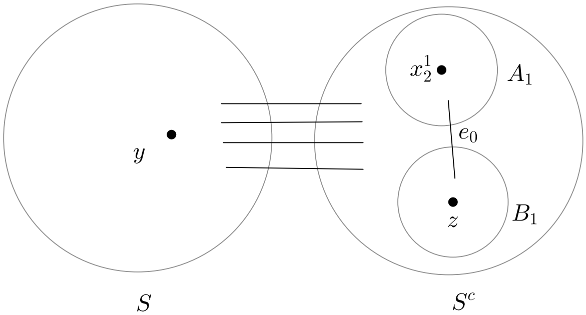


Fig. 3. The edge-cut $[S, S^c]_{G^*}$ of G^*.

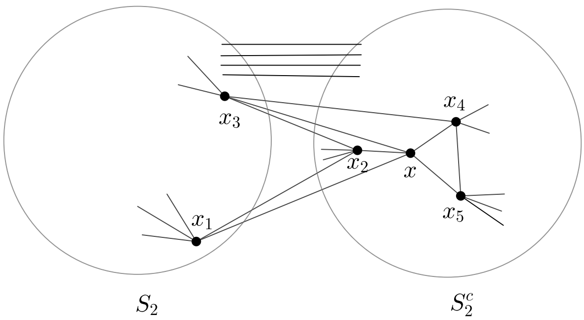


Fig. 4. The edge-cut $[S_2, S_2^c]_G$ of G.

Let $S_2 = (S - \{y\}) \cup \{x_1, x_3\}$. Clearly, $S_2 \cap S_1 = \emptyset$, $x \notin S_2$ and $d_G(S_2) = d_{G^*}(S) + |\{x_1x, x_1x_2, x_3x_2, x_3x, x_3x_4\}| \leq 9$, as shown in Fig. 4. So $G[S_2]$ is connected by Observation 3.6.

Second, we prove that $G[S_2] - e_0$ contains an (x_1, x_3)-path Q. Suppose it does not, and assume that A_2 and B_2 are two disjoint subsets of S_2 such that $G[A_2]$ and $G[B_2]$ are two components of $G[S_2] - e_0$ with $x_3 \in A_2$ and $x_1 \in B_2$. Since $e_0 \neq x_1x_3$, we have $|A_2| \geq 2$ or $|B_2| \geq 2$. WLOG, assume $|A_2| \geq 2$. It is clear that $d_G(A_2) \geq 8$ by Claim III and $d_G(B_2) \geq 5$ since G is 5-edge-connected. This is a contradiction to the fact that $d_G(A_2) + d_G(B_2) \leq d_G(S_2) + 2|\{e_0\}| \leq 11$. Hence, $G[S_2] - e_0$ contains an (x_1, x_3)-path Q.

Finally, we are now ready to derive a contradiction by planarity. Since $S_1 \cap S_2 = \emptyset$ and $x \notin S_1 \cup S_2$, we conclude that P and Q are two vertex-disjoint paths not through the vertex x and the edge e_0, contradicting the planarity of $G - e_0$. See Fig. 5 for an illustration of paths P and Q. $\square$

Claim VII. *Any modulo 3-orientation D^* of G^* can be extended to a modulo 3-orientation D of G.*

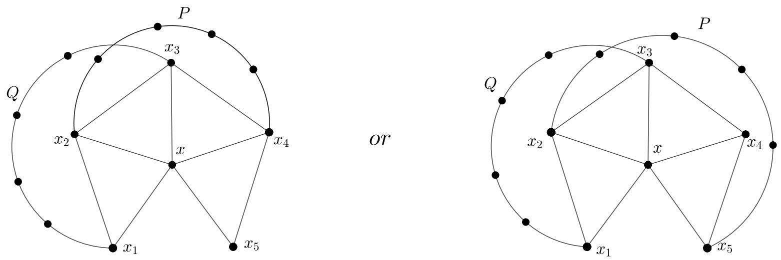


Fig. 5. The paths P and Q in $G - e_0$.

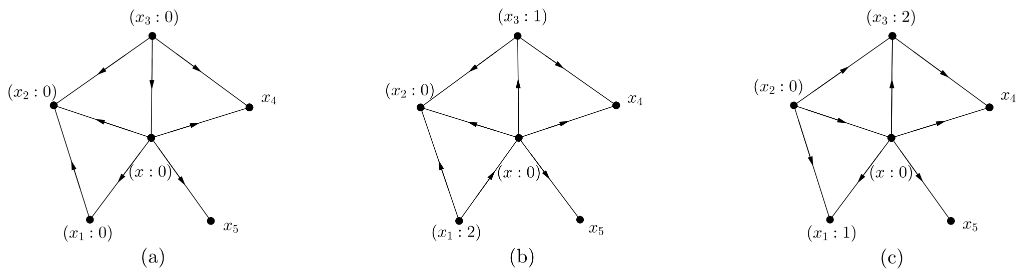


Fig. 6. Some orientations of the graph $H - x_4x_5$ in Claim VII.

Proof. Recall that the vertices and edges of G^* are labeled as in Fig. 1(c). Let D' be the restriction of D^* on G. Clearly, every vertex in $V(G) - V(H)$ is balanced modulo 3 in D'. We define $\beta(v) \equiv d^-_{D'}(v) - d^+_{D'}(v) \pmod 3$ for each $v \in V(H)$. Then it suffices to prove that H has a β-orientation D'' (that is, $d^+_{D''}(v) - d^-_{D''}(v) \equiv \beta(v) \pmod 3$ for each $v \in V(H)$). Note that we have

$$\beta(x_2) \equiv \beta(x) \equiv 0 \pmod 3,$$
$$\beta(x_1) + \beta(x_3) \equiv 0 \pmod 3, \text{ and } \beta(x_4) + \beta(x_5) \equiv 0 \pmod 3.$$

There are up to nine cases by considering $\beta(x_1) \in \{0, 1, 2\}$ and $\beta(x_5) \in \{0, 1, 2\}$. In Fig. 6, we demonstrate several orientations of $H - x_4x_5$, where we simply write $(v : i)$ for $\beta(v) \equiv i \pmod 3$. By appropriately orienting x_4x_5, the orientation in Fig. 6(a) and its reverse provide such β-orientation for $\beta(x_1) = 0$ and $\beta(x_5) \in \{0, 1, 2\}$. Similarly, after appropriately orienting x_4x_5, the orientation in Fig. 6(b) and its reverse offer such β-orientation for $\beta(x_1) = 2$ and $\beta(x_5) \in \{0, 1\}$ and for $\beta(x_1) = 1$ and $\beta(x_5) \in \{0, 2\}$. Moreover, after appropriately orienting x_4x_5, the orientation in Fig. 6(c) and its reverse give such β-orientation for $\beta(x_1) = 1$ and $\beta(x_5) \in \{0, 1\}$ and for $\beta(x_1) = 2$ and $\beta(x_5) \in \{0, 2\}$. This verifies all those nine cases and proves Claim VII. $\square$

5.3. The final step

Let G'' be a nearly-planar graph obtained from G^* by lifting $e^* = x_2^1v_1$ and $e^{**} = x_2^1v_2$, and deleting the vertex x_2^1. It is clear that G'' satisfies $\delta(G'') \geq 5$ and any essential

4^--edge-cut of G'' corresponds to an essential 4^--edge-cut of G^*. By Claim VI and Observation 3.3, we obtain that G'' is a 5-edge-connected nearly-planar graph with a handle-edge incident with u. Then we have a new pre-orientation D''_u on $E_{G''}(u)$ by restricting D_u on $E_{G''}(u)$ and setting $D''_u(v_jv_i) = D(x_2v_i)$ if x_2v_i is pre-oriented in G for $\{i, j\} = \{1, 2\}$. The minimality of G implies that G'' has an $\mathcal{M}_3$-extension of D''_u. Thus, by Observation 3.3(iii) and Claim VII, we obtain that G^* has an $\mathcal{M}_3$-extension of D_u and so does G, a contradiction. This completes the proof. □

6. Proof of Lemma 3.2 using Lemma 3.1

Finally, we are ready to finish the proof of Lemma 3.2, showing that every odd-5-edge-connected nearly-planar graph admits a modulo 3-orientation.

Suppose that G is a counterexample to Lemma 3.2 chosen with $|V(G)| + |E(G)|$ minimal. If $|V(G)| \leq 2$, then Lemma 3.2 holds trivially. Hence $|V(G)| \geq 3$. Assume $e_0 = uu_0$ is a handle-edge of G. Clearly, G is connected. Using Lemma 3.9 and Observation 3.4, with a similar argument as Claim II, we obtain that G is 2-connected and G contains no parallel edges. We embed the planar graph $G - e_0$ in the plane and still use $G - e_0$ to denote the planar embedding for notational convenience.

Next, we claim that G is 5-regular. For otherwise, there exists a vertex v with $d_G(v) \neq 5$. Similar as in Claim IV, we apply Lemma 4.1 to lift a pair of edges incident with v in a way that keeps near-planarity and does not create loops. The obtained new nearly-planar graph is denoted by G'. By Lemma 4.1, G' has no 1- or 3-edge-cut, and so G' has a modulo 3-orientation by the minimality of G. This implies that G has a modulo 3-orientation by Observation 3.3(iii), a contradiction.

Then we show that G is 4-edge-connected. Suppose to the contrary that G contains an essential 2-edge-cut $[S, S^c]_G$. WLOG, assume $u \in S$. Note that $e_0 \notin G[S^c]$. By Observation 3.4(ii), $G/G[S^c] \in \mathcal{N}$. Since G is a minimal counterexample, we get a modulo 3-orientation D_1 of $G/G[S^c]$. Then, in G, we contract $G[S]$ to a vertex u'. Since $G[S^c]$ is a planar graph and u' has degree 2 in $G/G[S]$, we have $G/G[S] \in \mathcal{N}$. Hence, $G/G[S]$ has a modulo 3-orientation D_2. Then either D_1 and D_2 agree along $[S, S^c]_G$ directly, or they agree after reversing all edge directions in D_2. Thus, their union provides a modulo 3-orientation of G, a contradiction.

By Lemma 3.1, G must contain an essential 4-edge-cut $[S, S^c]_G$. WLOG, assume $|S^c \cap \{u, u_0\}| \geq 1$. Let $T \subseteq S$ be a minimal subset of $V(G)$ such that $[T, T^c]_G$ is an essential 4-edge-cut. Clearly, $|T| \geq 2$ since G is 5-regular. By the minimality of T, for any proper subset $T_1 \subset T$ with $|T_1| \geq 2$, we have $d_G(T_1) \geq 5$. Note that $G[T]$ and $G[T^c]$ are both 2-edge-connected, since G is 4-edge-connected. Hence, by Observation 3.4(iii), $G/G[T] \in \mathcal{N}$ and $G/G[T^c] \in \mathcal{N}$.

Since $|S^c \cap \{u, u_0\}| \geq 1$ and $T \subseteq S$, we have $|T^c \cap \{u, u_0\}| \geq 1$. WLOG, assume $u \in T^c$. The minimality of G implies that $G/G[T]$ has a modulo 3-orientation D_1. Then we contract $G[T^c]$ to a vertex u', and we assign a pre-orientation $D'_{u'}$ which is the restriction of D_1 on $G/G[T^c]$. By Observation 3.4(ii), $G/G[T^c]$ is a planar graph if

$u_0 \in T^c$ and $G/G[T^c]$ is a nearly-planar graph with the handle edge e_0 incident with u' if $u_0 \in T$. By replacing an oriented edge incident with u' other than e_0 by two oriented edges in the opposite direction, we obtain a new 5-edge-connected nearly-planar graph G' and a new pre-orientation $D''_{u'}$ from $G/G[T^c]$ and $D'_{u'}$. By Lemma 3.1, G' has an $\mathcal{M}_3$-extension of $D''_{u'}$. Thus, $G/G[T^c]$ has an $\mathcal{M}_3$-extension of $D'_{u'}$ and G has a modulo 3-orientation, a contradiction. This completes the proof. □

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