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ORIGINAL PAPER

Spanning Triangle-Trees and Flows of Graphs

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Abstract
In this paper we study the flow properties of graphs containing a spanning triangle-tree. Our main results provide a structure characterization of graphs with a spanning triangle-tree admitting a nowhere-zero 3-flow. All these graphs without nowhere-zero 3-flows are constructed from K_4 by a so-called bull-growth operation. This generalizes a result of Fan et al. in 2008 on triangularly-connected graphs and particularly shows that every 4-edge-connected graph with a spanning triangle-tree has a nowhere-zero 3-flow. A well-known classical theorem of Jaeger in 1979 shows that every graph with two edge-disjoint spanning trees admits a nowhere-zero 4-flow. We prove that every graph with two edge-disjoint spanning triangle-trees has a flow strictly less than 3.

Keywords Nowhere-zero flow · 3-Flow flow index · Triangularly-connected · Triangle-tree · 2-Tree

Mathematics Subject Classification 05C21 · 05C40 · 05C05

1 Introduction

We shall introduce some necessary notation and terminology and the concepts of 3-flows, circular flows and group connectivity in the next subsections.

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1.1 Nowhere-Zero 3-Flows

Graphs considered here may contain parallel edges, but no loops. We follow the textbook [3] for undefined terminology and notation. For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. When S is an edge subset of $E(G)$ or a vertex subset of $V(G)$, we use $G[S]$ to denote the edge-induced subgraph or the vertex-induced subgraph from S. For a vertex $u \in V(G)$, $d_G(u)$ denotes the degree of u in G. Sometimes the subscript is omitted for convenience. We call u a *k-vertex* (*k^+-vertex*, resp.) if $d(u) = k$ ($d(u) \geq k$, resp.). A *k-cut* is an edge-cut of size k. Let D be an orientation of G. The set of outgoing arcs incident to u is denoted by $E_D^+(u)$, while the set of incoming arcs is denoted by $E_D^-(u)$. We use $d_D^+(u) = |E_D^+(u)|$ and $d_D^-(u) = |E_D^-(u)|$ to denote the out-degree and in-degree of u, respectively.

Given an orientation D and a function f from $E(G)$ to $\{\pm 1, \pm 2, \ldots, \pm(k-1)\}$, if $\sum_{e \in E_D^+(v)} f(e) = \sum_{e \in E_D^-(v)} f(e)$ for each vertex $v \in V(G)$, then we call (D, f) a *nowhere-zero k-flow*, abbreviated as *k-NZF*. The flow theory was initiated by Tutte [21], generalizing face-colorings of plane graphs to flows of arbitrary graphs by duality. Tutte proposed the well-known 3-flow conjecture, which was selected by Bondy among the *Beautiful Conjectures in Graph Theory* [2] with high evaluation.

Conjecture 1.1 (*Tutte's 3-flow conjecture*) *Every 4-edge-connected graph has a 3-NZF.*

Jaeger's 4-flow theorem [8] from 1979 shows that every 4-edge-connected graph admits a nowhere-zero 4-flow. This theorem was proved by finding even subgraph covers from spanning trees, and a stronger version concerning spanning trees is as follows.

Theorem 1.1 [8] *Every graph with two edge-disjoint spanning trees has a 4-NZF.*

For graphs with higher edge-connectivity, breakthrough results for Conjecture 1.1 were obtained by Thomassen [20] and Lovász et al. [18], which eventually confirmed Conjecture 1.1 for 6-edge-connected graphs.

Theorem 1.2 [18] *Every 6-edge-connected graph admits a 3-NZF.*

On the other hand, Kochol [11] proved that it suffices to prove Conjecture 1.1 for 5-edge-connected graphs and he also showed that Conjecture 1.1 is equivalent to the statement that every bridgeless graph with at most three 3-cuts admits a 3-NZF. There are infinitely many graphs with exactly four 3-cuts but admitting no 3-NZF. Several such graph families were given in [5, 12, 13]. Most of these graphs consist of 2-sums of K_4 (defined later), and majority of their edges lie in triangles. This may suggest that the potential minimal counterexamples to Conjecture 1.1 (or its equivalent form) may contain many triangles. For more examples, see [4] which characterizes all planar non vertex-3-colorable graphs with four triangles, whose duals also contain similar structures involving four 3-cuts and many triangles.

A graph is *triangular* if each edge is contained in a triangle K_3. Xu and Zhang [22] suggested to consider Conjecture 1.1 for triangular graphs and they verified

Conjecture 1.1 for squares of graphs, a subclass of triangular graphs. Other examples of triangular graphs are the triangulations on surfaces, chordal graphs and locally connected graphs, whose flow properties were studied in [1, 12, 13], among others.

Definition 1.1 A *triangle-tree* $\mathcal{T}(x_1, x_2, \ldots, x_n)$ is formed by starting with a triangle $x_1x_2x_3$ and then repeatedly adding vertices in such a way that each added vertex x_{j+1} is connected to exactly two adjacent vertices y, z in $\mathcal{T}(x_1, x_2, \ldots, x_j)$. Note that the vertices x_{j+1}, y, z exactly form a triangle. A 2-vertex in the triangle-tree is called a *leaf*. For $n \geq 4$, a *triangle-path* $\mathcal{P}(x_1, x_2, \ldots, x_n)$ is a triangle-tree with precisely two leaves. In the trivial case $n = 3$, $\mathcal{P}(x_1, x_2, x_3)$ is a triangle, also considered as a trivial triangle-path.

A graph G is *triangularly-connected* if for any pair of edges $e_1, e_2 \in E(G)$, there is a triangle-path containing e_1 and e_2.

The above-mentioned graph classes presented in [1, 12, 13, 22] are all triangularly-connected. Fan et al. [5] obtained a complete characterization of triangularly-connected graphs with 3-NZF using 2-sum operations. Let A, B be two subgraphs of G. We call G the 2-*sum* of A and B, denoted by $G = A\bigoplus_2 B$, if $E(G) = E(A)\bigcup E(B)$, $|E(A)\bigcap E(B)| = 1$ and $|V(A)\bigcap V(B)| = 2$. The wheel graph W_k is constructed by adding a center vertex connected to each vertex of a k-cycle, where $k \geq 3$. A wheel W_k is odd (even, resp.) if k is an odd (even, resp.) number. Note that K_4 is also viewed as the odd wheel W_3.

Theorem 1.3 (*Fan et al.* [5]) *Let G be a triangularly-connected graph. Then G has no* 3*-NZF if and only if there is an odd wheel W and a subgraph G_1 such that $G = W\bigoplus_2 G_1$, where G_1 is a triangularly-connected graph without* 3*-NZF.*

In this paper, we push further to study a related graph class, i.e., graphs containing a spanning triangle-tree. Triangularly-connected graphs may contain a spanning triangle-tree, but graphs containing a spanning triangle-tree may not be triangularly-connected, see Figs. 2 and 3 for instance. More detailed comparison of these two graph classes is discussed in the last section. In fact, our main results hold for a wider graph class (i.e., graphs containing a spanning triangularly-connected subgraph), which includes both graphs containing a spanning triangle-tree and triangularly-connected graphs. See Theorem 5.2 for more details.

In our characterization, we need to handle certain 3-connected graphs, and the 2-sum operation is not sufficient for this work. Thus we develop a new tool, called the *bull-growth/bull-reduction*.

Definition 1.2 Let u, v be two adjacent 3-vertices of a graph G with a common neighbor w. The third neighbor of u and v is denoted by a and b, respectively. Let $G_1 = G - u - v + ab$ (and we delete possible loops when $a = b$). Then G_1 is called the *bull-reduction* of G, and G is a *bull-growth* of G_1 (see Fig. 1), and we write $G = \mathcal{B} \uplus G_1$.

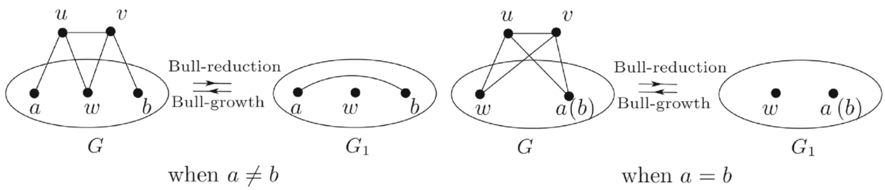

Fig. 1 Bull-reduction and bull-growth

Theorem 1.4 *Let G be a graph containing a spanning triangle-tree. Then G has no 3-NZF if and only if* $G = \mathcal{B} \biguplus G_1$, *where* G_1 *contains a spanning triangle-tree and has no 3-NZF. In other words, G has no 3-NZF if and only if G is formed from* K_4 *by a series of bull-growth operations.*

Since each step of the bull-growth operation on a graph does not decrease the number of 3-vertices in the graph, we obtain a direct corollary of Theorem 1.4, verifying Conjecture 1.1 for those graphs in a strong sense.

Corollary 1.1 *Every graph with a spanning triangle-tree and with at most three 3-vertices has a 3-NZF.*

1.2 Circular Flows and Group Connectivity

For integers $t \geq 2s > 0$, a circular t/s-flow of a graph G is a t-NZF (D, f) such that $s \leq |f(e)| \leq t - s$ for any edge $e \in E(G)$. The flow index was defined in [6] as the least rational number r such that G has a circular r-flow. Jaeger [9] generalized Tutte's flow conjectures and proposed a conjecture that every $4k$-edge-connected graph admits a circular $(2 + 1/k)$-flow. It was confirmed for $6k$-edge-connected graph by Lovász et al. [18], while eventually disproved in [7] for $k \geq 3$. But the cases for $k = 1, 2$ concerning 4-, 8-edge-connected graphs are still particularly important since they imply Tutte's 3-flow and 5-flow conjectures, respectively. Closely related to those conjectures, the authors in [17] studied the problem of flow index less than 3, sandwiched between 2.5 and 3. They proved that every 8-edge-connected graph has a flow index strictly less than 3, and conjectured that 6-edge-connectivity suffices. Here we obtain a result for the flow index less than 3 in the spirit of Theorem 1.1.

Theorem 1.5 *Every graph with two edge-disjoint spanning triangle-trees has a flow index strictly less than* 3.

Almost of all the above-mentioned flow results in fact use some orientation techniques. An orientation D of G is a *mod k-orientation* if for each vertex v of $V(G)$, $d_D^+(v) - d_D^-(v) = 0(\text{mod k})$. The study of 3-flows frequently uses mod 3-orientation, since Tutte [21] proved that a graph has a 3-NZF if and only if it admits a mod 3-orientation. This fact was generalized by Jaeger [9] who showed that a graph has a circular $(2 + 1/p)$-flow if and only if it admits a mod $(2p + 1)$-orientation. Moreover, it was proved in [17] that a connected graph has a flow index strictly less than $2 + 1/p$ if and only if it admits a strongly connected mod $(2p + 1)$-

orientation. Hence, we shall prove Theorem 1.5 using strongly connected mod 3-orientations.

Serving for a stronger induction process in proof, we will sometimes need certain orientation with prescribed boundaries, that is the concept of *group connectivity* introduced by Jaeger et al. [10]. For more on the group connectivity, we refer to [15]. A $\mathbb{Z}_3$-*boundary* β of a graph G is a mapping from $V(G)$ to $\mathbb{Z}_3$ with $\sum_{v\in V(G)} \beta(v) \equiv 0 (\text{mod } 3)$. If for any $\mathbb{Z}_3$-boundary β, there is an orientation D of G such that $d_D^+(v) - d_D^-(v) \equiv \beta(v) (\text{mod } 3)$ for any vertex $v \in V(G)$, then we say that G is $\mathbb{Z}_3$-*connected*. Denote by $\langle \mathbb{Z}_3 \rangle$ the set of all the $\mathbb{Z}_3$-connected graphs. The advantage of this stronger property is to allow us to extend a mod 3-orientation of G/H to that of G when the subgraph H is $\mathbb{Z}_3$-connected (cf. [10, 12, 18]). For strongly connected mod 3-orientations, a similar property is defined in [17]. Let $\mathcal{S}_3$ be the family of all graphs G such that for any $\mathbb{Z}_3$-boundary β, there is a strongly connected orientation D of G satisfying that $d_D^+(u) - d_D^-(u) \equiv \beta(u) (\text{mod } 3), \forall u \in V(G)$. In fact, a stronger form of Theorem 1.5 is proved in Sect. 4 that for any graph G with $|V(G)| \geq 4$ containing two edge-disjoint spanning triangle-trees, we have $G \in \mathcal{S}_3$.

Jaeger et al. [10] proposed a conjecture, strengthening Conjecture 1.1, that every 5-edge-connected graph is $\mathbb{Z}_3$-connected. Theorem 1.3 above also has a form on $\mathbb{Z}_3$-group connectivity in Fan et al. [5]: for any triangularly-connected graph G, $G \notin \langle \mathbb{Z}_3 \rangle$ if and only if G is constructed from 2-sums of triangles and odd wheels(see Theorem 5.1 in Sect. 5). Our $\mathbb{Z}_3$-group connectivity version of Theorem 1.4 has a similar feature, with additional bull-growth operations.

Theorem 1.6 *Let G be a graph with a spanning triangle-tree. Then $G \notin \langle \mathbb{Z}_3 \rangle$ if and only if G can be constructed by one of the following operations*:

(i) G is K_3 or K_4.
(ii) $G = K_3 \bigoplus_2 G_1$, where $G_1 \notin \langle \mathbb{Z}_3 \rangle$ contains a spanning triangle-tree.
(iii) $G = \mathcal{B} \uplus H$, where $H \notin \langle \mathbb{Z}_3 \rangle$ contains a spanning triangle-tree.

Theorem 1.6 also verifies the the special case of the conjecture of Jaeger et al. [10] for $\mathbb{Z}_3$-connectedness on 4-edge-connected graphs containing a spanning triangle-tree.

A *crystal* is a graph consisting of a triangle-path plus an extra edge connecting two leaves of the triangle-path. For instance, a wheel is a crystal by definition, and some more examples are depicted in Fig. 3. Crystals are special graphs containing a spanning triangle-tree, and also play a role in our proofs. We obtain the following characterization of crystals as corollaries of Theorems 1.4 and 1.6, connecting flows and vertex-coloring of crystals.

Corollary 1.2

(i) A crystal has no 3-NZF if and only if every vertex is of odd degree.
(ii) A crystal is $\mathbb{Z}_3$-connected if and only if it is vertex-3-colorable.

2 Basic Lemmas and Bull-Growth Operation

We start with some basic lemmas, most of which have been widely used in flow theory. The following complete family properties were obtained in [12] for $\langle \mathbb{Z}_3 \rangle$ and in [17] for $\mathcal{S}_3$. Here we use sK_2 to denote the graph with two vertices and s parallel edges.

Lemma 2.1 [12] [17] *Let $\mathcal{F} \in \{\langle \mathbb{Z}_3 \rangle, \mathcal{S}_3\}$. Then each of the following holds.*

(i) $K_1 \in \mathcal{F}$.
(ii) If $e \in E(G)$ and $G \in \mathcal{F}$, then $G/e \in \mathcal{F}$.
(iii) If $H, G/H \in \mathcal{F}$, then $G \in \mathcal{F}$.
(iv) $2K_2 \in \langle \mathbb{Z}_3 \rangle$ and $4K_2 \in \mathcal{S}_3$.

The lifting lemma below on flows is routine to verify by definitions, as observed in [14, 16]. When $va, vb \in E_G(v)$, let $G_{[v,ab]} = G - va - vb + ab$ denote the graph obtained from G by lifting va, vb to become ab.

Lemma 2.2 [14] [16] *Let v be a 4^+-vertex of a graph G with $va, vb \in E_G(v)$.*

(i) If $G_{[v,ab]} \in \langle \mathbb{Z}_3 \rangle$, then $G \in \langle \mathbb{Z}_3 \rangle$.
(ii) If $G_{[v,ab]}$ has a 3-NZF, then so does G.
(iii) If $G_{[v,ab]} \in \mathcal{S}_3$, then so does G.
(iv) If $G - v + ab \in \mathcal{S}_3$, then so does G.

By repeatedly applying Lemma 2.2(i), we immediately obtain the following more general lifting lemma, which will be a useful tool in our proofs.

Lemma 2.3 *Let P be a path from u to v in G. If $G - E(P) + uv \in \langle \mathbb{Z}_3 \rangle$, then $G \in \langle \mathbb{Z}_3 \rangle$.*

We refer to this operation as *lifting E(P) in G to become a new edge uv*.

In a tree T, for any $u, v \in V(T)$ there is a unique uv-path from u to v, denoted by P_{uv}. A uwv-path means a path from u to v which goes through w, denoted by P_{uwv}. Fix a triangle-tree $\mathcal{T}$ and let $x, y \in V(\mathcal{T}) \cup E(\mathcal{T})$ be two nonadjacent elements. There is a shortest xy-triangle-path from x to y, denoted by $\mathcal{P}(x, y, \mathcal{T})$. That is a sequence of triangles $R_1, R_2, \ldots, R_s$ from x to y with $|E(R_i) \cap E(R_{i+1})| = 1$ and $|E(R_i) \cap E(R_{j+1})| = 0$ for $1 \le i \le s-1$ and $j > i+1$. We write $\mathcal{P}(x, y)$ for convenience if no confusion occurs.

Lemma 2.4 *Let G be a graph containing a spanning triangle-tree $\mathcal{T} = \mathcal{T}(x_1, x_2, \ldots, x_n)$, where x_1 is a leaf of $\mathcal{T}$.*

(i) For any $j, k > 1$, the graph $\mathcal{T} + x_1x_j + x_1x_k$ is $\mathbb{Z}_3$-connected.
(ii) Let $u, v, w \in V(\mathcal{T})$. If $w \notin V(\mathcal{P}(u, v, \mathcal{T}))$, then the graph $\mathcal{T} + uw + vw$ is $\mathbb{Z}_3$-connected.
(iii) If $G - \mathcal{T}$ contains a cycle, then $G \in \langle \mathbb{Z}_3 \rangle$.

Proof

(i) Since $x_1x_2x_3$ is a triangle in $H = \mathcal{T} + x_1x_j + x_1x_k$, we lift x_1x_2, x_1x_3 to obtain a graph $H_{[x_1,x_2x_3]}$ which contains parallel edges x_2x_3. Applying Lemma 2.1(iii), (iv) to contract 2-cycles consecutively along $\mathcal{T} - x_1$, we obtain a $2K_2 \in \langle \mathbb{Z}_3 \rangle$ which consists of the edges x_1x_j, x_1x_k. Hence, $H_{[x_1,x_2x_3]} \in \langle \mathbb{Z}_3 \rangle$, and so $H \in \langle \mathbb{Z}_3 \rangle$ by Lemma 2.2(i).

(ii) Since w is not in $\mathcal{P}(u, v, \mathcal{T})$, in $\mathcal{T}$ there is a shortest triangle-path $\mathcal{P}$ from w to an edge in $\mathcal{P}(u, v, \mathcal{T})$ among all possible choices. Then $\mathcal{P}(u, v, \mathcal{T}) \cup \mathcal{P}$ is a triangle-tree, where w is a leaf of it. Set $H = \mathcal{P}(u, v, \mathcal{T}) \cup \mathcal{P} + uw + vw$. Then $H \in \langle \mathbb{Z}_3 \rangle$ by Lemma 2.4(i). Note that H is a subgraph of $T + uw + vw$. In $\mathcal{T} + uw + vw$, we contract H and then contract the resulting 2-cycles consecutively. Eventually we get a K_1. Hence $\mathcal{T} + uw + vw \in \langle \mathbb{Z}_3 \rangle$ by Lemma 2.1(iii). Note that the Lemma also holds when $u = v$, in which case we can choose any triangle containing u as $\mathcal{P}(u, v, \mathcal{T})$.

(iii) Let C be a cycle of $G - \mathcal{T}$. If $V(C) = 2$, there is a 2-cycle uw of G. Then Lemma 2.4(ii) is applied with $u = v$, and so $H_1 = \mathcal{T} + uw + uw$ is $\mathbb{Z}_3$-connected. As both H_1 and G/H_1 are $\mathbb{Z}_3$-connected, we have $G \in \mathbb{Z}_3$ by Lemma 2.1(iii).

If $V(C) \geq 3$, suppose $u, v, w \in V(C)$, and $E(C)$ consists of three edge-disjoint paths P_{uv}, P_{vw}, P_{wu} in the cyclic order. There is a triangle-path $\mathcal{P}(u, v, \mathcal{T})$ since $\mathcal{T}$ is a spanning triangle-tree. If $w \notin V(\mathcal{P}(u, v, \mathcal{T}))$, then we lift P_{vw}, P_{wu} to become two edges vw, uw, and $\mathcal{T} + vw + uw \in \langle \mathbb{Z}_3 \rangle$ by Lemma 2.4(ii). Thus, $G \in \langle \mathbb{Z}_3 \rangle$ by Lemma 2.3. If $w \in V(\mathcal{P}(u, v, \mathcal{T}))$, then we must have $u \notin V(\mathcal{P}(w, v, \mathcal{T}))$. In this case we lift P_{vu}, P_{wu} to become two edges vu, wu. Hence, $\mathcal{T} + vu + wu \in \langle \mathbb{Z}_3 \rangle$ by Lemma 2.4(ii), and so $G \in \langle \mathbb{Z}_3 \rangle$ by Lemma 2.1 again. □

Note that, if any added edges in Lemma 2.4(i) and (ii) are replaced by corresponding paths connecting the end vertices, we get $\mathbb{Z}_3$-connected graphs by Lemma 2.3. From Lemma 2.4, we also obtain the following corollary by applying Lemma 2.1 to contract $\mathbb{Z}_3$-connected subgraphs.

Corollary 2.1 *Let G be a graph with a spanning triangle-tree. Then $G \in \langle \mathbb{Z}_3 \rangle$ if and only if it contains a nontrivial $\mathbb{Z}_3$-connected subgraph.*

Proof Let H be a nontrivial $\mathbb{Z}_3$-connected subgraph and $\mathcal{T}$ a spanning triangle-tree of G. If $E(\mathcal{T}) \cap E(H) \neq \emptyset$, then in G we contract the $\mathbb{Z}_3$-connected subgraph H and then repeatedly contract 2-cycles to eventually get a singleton K_1. Thus, $G \in \langle \mathbb{Z}_3 \rangle$ by Lemma 2.1(iii). Otherwise, $E(\mathcal{T}) \cap E(H) = \emptyset$. Since a $\mathbb{Z}_3$-connected graph must be 2-edge-connected, H contains a cycle which is edge-disjoint with the spanning triangle-tree $\mathcal{T}$ of G. Hence, $G \in \langle \mathbb{Z}_3 \rangle$ by Lemma 2.4(iii). □

Now we present the bull-growth operation as a key tool in our later proofs.

Lemma 2.5 *Let $G = \mathcal{B} \uplus G_1$. The following statements hold.*

(i) G has a 3-NZF if and only if G_1 has a 3-NZF.

(ii) If $G \in \langle \mathbb{Z}_3 \rangle$, then $G_1 \in \langle \mathbb{Z}_3 \rangle$. Conversely, if $G_1 \notin \langle \mathbb{Z}_3 \rangle$, then $G \notin \langle \mathbb{Z}_3 \rangle$.

Proof We adopt the notation as in Definition 1.2. Let $G_1 = G - u - v + ab$, where u, v are two adjacent 3-vertices with a common neighbor w. We shall verify (ii) first, and we show $G_1 \in \langle \mathbb{Z}_3 \rangle$ by definition.

Let β_1 be a $\mathbb{Z}_3$-boundary of G_1. Define $\beta : V(G) \to \mathbb{Z}_3$ as follows:

$$\begin{cases} \beta(u) = \beta(v) = 0, \\ \beta(x) = \beta_1(x), \forall x \notin \{u, v\}. \end{cases}$$

Since $\sum_{t \in V(G)} \beta(t) = \sum_{x \in V(G_1)} \beta_1(x) \equiv 0 (\text{mod } 3)$, β is a $\mathbb{Z}_3$-boundary of G. As $G \in \langle \mathbb{Z}_3 \rangle$, G has an orientation D such that $d_D^+(x) - d_D^-(x) \equiv \beta(x) (\text{mod } 3), \forall x \in V(G)$. Since $\beta(u) = \beta(v) = 0$ and u, v are adjacent, one of u, v is oriented as all ingoing and the other is oriented as all outgoing. Thus uw and vw receive opposite orientations in D. Moreover, the edges au, vb are either oriented from a to u and from v to b, or all receive opposite directions. So, we can orient ab the same as au and keep the orientations of the other edges of G_1 the same as D. Then this gives an orientation D_1 of G_1 with $d_{D_1}^+(y) - d_{D_1}^-(y) \equiv \beta_1(y) (\text{mod } 3), \forall y \in V(G_1)$. So, $G_1 \in \langle \mathbb{Z}_3 \rangle$ by definition.

Recall that a graph has a 3-NZF if and only if it has a mod 3-orientation. Thus (i) follows from a similar argument with (ii) by replacing β_1-boundary with a zero-boundary. One may also see that the path $auvb$ of G plays the same role as the edge ab of G_1 in a mod 3-orientation and the process can be reversed as well. □

The reverse of Lemma 2.5 (ii) is not true in general, for example, it fails when G_1 is an odd wheel (and $a \neq b$ in bull-growth). However, when G contains a spanning triangle-tree, Lemma 2.5 can be strengthened as follows, which becomes a necessary and sufficient statement.

Lemma 2.6 *Let G be a graph with a spanning triangle-tree and $G = B \uplus G_1$. Then $G \in \langle \mathbb{Z}_3 \rangle$ if and only if $G_1 \in \langle \mathbb{Z}_3 \rangle$.*

Proof We still adopt the same notation as above and let $G_1 = G - u - v + ab$. Since G has a spanning triangle-tree $\mathcal{T}$, at least one of the edges of $\mathcal{T}$ must be in $\{aw, bw\}$, say $bw \in E(\mathcal{T})$. We will show below that $G_1 \in \langle \mathbb{Z}_3 \rangle$ implies $G \in \langle \mathbb{Z}_3 \rangle$.

Let $\beta : V(G) \to \mathbb{Z}_3$ be a $\mathbb{Z}_3$-boundary of G. If $\beta(u) \neq 0$, we lift uw, uv to become a new edge vw, and then delete the vertex u. Let H be the resulting graph with corresponding boundary β_1, where $\beta_1(a) = \beta(a) + \beta(u)$ and $\beta_1(z) = \beta(z), \forall z \in V(G) \backslash \{u, a\}$. Then H contains a $\mathbb{Z}_3$-connected subgraph $2K_2$ which consists of two parallel edges vw. By Corollary 2.1, we have $H \in \langle \mathbb{Z}_3 \rangle$, and so H has an orientation D_1 satisfying boundary β_1. We orient ua to satisfy $\beta(u)$ and add vu, uw back with their orientations kept as the lifted edge vw of D_1. Specifically, we orient ua from u to a if $\beta(u) = 1$, and orient it from a to u if $\beta(u) = -1$. This provides an orientation of G satisfying boundary β.

If $\beta(v) \neq 0$, an analogous argument applies. We lift vb, vw to become a new edge bw and delete the vertex v. Let H be the resulting graph with corresponding boundary β_1 defined similarly as above. Then the resulting graph H is in $\langle \mathbb{Z}_3 \rangle$ and the β_1-orientation of H can be extended to G by imitating the proof above.

If $\beta(u) = \beta(v) = 0$, we define a $\mathbb{Z}_3$-boundary β_1 of G_1 as $\beta_1(x) = \beta(x)$ for any $x \in V(G) \setminus \{u, v\}$. Since $G_1 \in \langle \mathbb{Z}_3 \rangle$, there is an orientation D_1 of G_1 satisfying β_1, where we may assume that the edge ab is oriented from a to b (the other case is similar). Then, in G we keep the orientation of $E(G_1) - ab$ as in D_1, and orient the rest of edges as all ingoing to u and outgoing to v. This gives an orientation of G satisfying boundary β as well. Therefore, G is $\mathbb{Z}_3$-connected by definition. □

Note that in the bull-reduction operation, the condition that G has a spanning triangle-tree $\mathcal{T}$ cannot ensure that G_1 contains a spanning triangle-tree. But if u or v is a leaf of $\mathcal{T}$, then the bull-reduction results in that G_1 contains a spanning triangle-tree. In the proof below, we shall always apply this operation for leaves of spanning triangle-trees implicitly.

Lemma 2.7 [5] *Let* $G = H_1 \bigoplus_2 H_2$.

(i) If $H_1 \notin \langle \mathbb{Z}_3 \rangle$ and $H_2 \notin \langle \mathbb{Z}_3 \rangle$, then $G \notin \langle \mathbb{Z}_3 \rangle$.
(ii) If neither H_1 nor H_2 has a 3-NZF, then G does not have a 3-NZF.

3 Graphs with a Spanning Triangle-Tree

Now we are ready to prove our main results, Theorems 1.6 and 1.4, for graphs containing a spanning triangle-tree.

Proof of Theorem 1.6: If G satisfies one of (i), (ii) and (iii), then $G \notin \langle \mathbb{Z}_3 \rangle$ by Lemmas 2.6 and 2.7. Now suppose that G satisfies none of (i),(ii) or (iii). We shall show that $G \in \langle \mathbb{Z}_3 \rangle$ by contradiction. Let G be a minimum counterexample of Theorem 1.6 with respect to $|E(G)| + |V(G)|$. Let $\mathcal{T}$ be a spanning triangle-tree of G. It is clear that for any vertex $v \in V(G)$, $d(v) \geq 3$. Otherwise, G satisfies condition (i) or (ii). To see this, we observe that a vertex v with $d(v) = 2$ is exactly a leaf of $\mathcal{T}$. So $G = K_3 \bigoplus_2 G_1$, where G_1 contains a spanning triangle-tree $\mathcal{T} - v$. By Corollary 2.1, we have $G_1 \notin \langle \mathbb{Z}_3 \rangle$, and thus condition (ii) holds.

Suppose $\mathcal{P} = \mathcal{P}(u, v)$ is a longest triangle-path among all possible triangle-paths in G. Let a, b be the neighbors of u on $\mathcal{P}$, where a is a vertex with exactly 3 neighbors in $\mathcal{P}$.

We first claim that

$$E(\mathcal{T}) \cap E(\mathcal{P}) \neq \emptyset. \tag{1}$$

It is clear that $\mathcal{P}$ contains a cycle. If no edge of $\mathcal{P}$ is in $E(\mathcal{T})$, then by Lemma 2.4(iii) we have $G \in \langle \mathbb{Z}_3 \rangle$. So, there is an edge of $\mathcal{P}$ in $E(\mathcal{T})$, and (1) holds.

Thus, for any vertex $t \in V(G) \setminus V(\mathcal{P})$, there is a triangle-path $\mathcal{P}(t, e)$ from t to some $e \in E(\mathcal{P})$ by (1). Denote by $\mathcal{P}(t, e_t)$ the shortest one among all triangle-paths $\mathcal{P}(t, e)$ with $e \in E(\mathcal{P})$. Note that $e_t \notin \{ua, ub\}$; otherwise, there is a longer triangle-path in G. If $t \in V(\mathcal{P})$, we also define $e_t = \emptyset$ and $\mathcal{P}(t, e_t) = \emptyset$ for technical reasons.

Next, we show the following statement:

$$d_G(u) = 3 \text{ and } u \text{ is a leaf of } \mathcal{T}. \tag{2}$$

Since G does not satisfy (i) and (ii), $d_G(u) \geq 3$. Suppose, by contradiction, that $d_G(u) \geq 4$, and s, d are two neighbors of u other than a, b. Let $H = \mathcal{P} \cup \mathcal{P}(s, e_s) \cup \mathcal{P}(d, e_d)$. Then H is a triangle-tree, and moreover, u is a leaf of H. Thus, $H + us + ud \in \langle \mathbb{Z}_3 \rangle$ by Lemma 2.4(i), and so $G \in \langle \mathbb{Z}_3 \rangle$ by Corollary 2.1, which is a contradiction. So, $d_G(u) = 3$ and u is a leaf of $\mathcal{T}$ by Definition 1.1 and the fact that $\mathcal{P} = \mathcal{P}(u, v)$ is the longest triangle-path in G. This proves (2).

Let x be the third neighbor of u, other than a, b. Let $\mathcal{Q} = \mathcal{P}(x, e_x)$ and c be third neighbor of a on $\mathcal{P}$, other than u, b. Then we have $e_x \notin \{ab, ac\}$. Otherwise, there is a longer triangle-path of G.

Let $G' = G_{[a,bc]} = G - ab - ac + bc$, and let H be a maximum $\langle \mathbb{Z}_3 \rangle$-subgraph of G' containing bc. Since bc is a 2-cycle, by Lemma 2.1(iii) we contract 2-cycles consecutively to obtain that $G'[V(\mathcal{P} \cup \mathcal{Q}) - a] \in \langle \mathbb{Z}_3 \rangle$, and so

$$V(\mathcal{P} \cup \mathcal{Q}) - a \subset V(H).$$

If $d_G(a) = 3$, then by (2) the bull-reduction in (iii) is applied for G, and the resulting graph has a spanning triangle-tree, a contradiction. Hence, $d_G(a) \geq 4$. Now we claim that

$$\text{there is a neighbor } y \text{ of } a \text{ that is not in } V(H). \tag{3}$$

Since $d_G(a) \geq 4$ and a has exactly 3 neighbors in $\mathcal{P}$, we may let y be a neighbor of a not in $V(\mathcal{P})$. If $y \in V(H)$, then there are at least two neighbors of a, namely u and y, in $V(H)$. By the maximality of H and Lemma 2.1(iii), (iv), we have $y \in V(H)$. Thus by Lemma 2.1(iii) again, it follows from $u, y \in V(H)$ that $a \in V(H)$. Now we conclude that $V(\mathcal{P} \cup \mathcal{Q}) \subset V(H)$. Applying Lemma 2.2(i), we also have $G[V(H)] \in \langle \mathbb{Z}_3 \rangle$, and so $G \in \langle \mathbb{Z}_3 \rangle$ by Corollary 2.1, a contradiction. This verifies (3).

Since $d_G(y) \geq 3$ and by Lemma 2.1(iii), at most one neighbor of y is in $V(H)$, and so there is a neighbor z of y not in $V(H)$. This also means that $\mathcal{P}(z, e_z)$ must intersect $\mathcal{P}$ at ab or ac, w.l.o.g., say $e_z = ac$. Otherwise, we have $z \in V(H)$, and so $y \in V(H)$ by Lemma 2.1(iii), a contradiction.

The final step If $\mathcal{P}(z, e_z)$ is a triangle acz, see Fig. 2, then $\mathcal{P} - u + za + zc + ya + yz$ is a longer triangle-path of G, a contradiction. Otherwise, $\mathcal{P}(z, e_z)$ contains

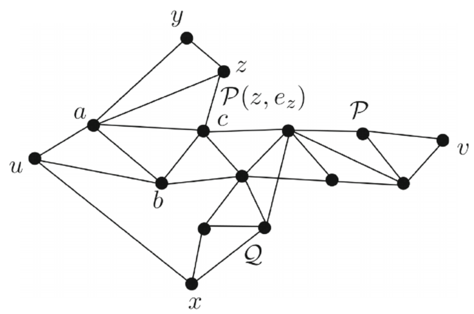

Fig. 2 A longer triangle-path

at least two triangles, and so $\mathcal{P} - u + \mathcal{P}(z, e_z)$ is a triangle-path longer than $\mathcal{P}$, again a contradiction to the maximality of $\mathcal{P}$. This finishes the proof. □

Proof of Theorem 1.4 If G is formed from K_4 by a series of bull-growth operations, then it has no 3-NZF by Lemma 2.5. Conversely, assume that G has no 3-NZF. Then, $G \notin \langle \mathbb{Z}_3 \rangle$. We apply Theorem 1.6 on G.

Suppose $G = K_3 \bigoplus_2 G_1$, where G_1 contains a spanning triangle-tree $\mathcal{T}$. Let abc correspond to the K_3 in the 2-sum, where a is a 2-vertex of G. Then $G_{[a,bc]}$ contains a 2-cycle bc, which shows $G_{[a,bc]} \in \langle \mathbb{Z}_3 \rangle$ by Corollary 2.1, and therefore, has a 3-NZF. Hence G has a 3-NZF by Lemma 2.2(ii), a contradiction.

Now suppose $G = \mathcal{B} \uplus G_1$, where G_1 contains a spanning triangle-tree $\mathcal{T}$. By Lemma 2.5, G_1 has no 3-NZF if and only if G has no 3-NZF. This proves Theorem 1.4. □

Proof of Corollary 1.2 Let $\mathcal{C} = \mathcal{P}(u, v) + uv$ be a crystal, where the vertices of $\mathcal{C}$ are ordered as $u, x_1, x_2, \ldots, x_k, v$ and $d_{\mathcal{C}}(x_1) = 3$ according to Definition 1.1(see Fig. 3). When $|V(\mathcal{C})| \leq 5$, $\mathcal{C}$ is a wheel and the statements clearly hold. Now we proceed by induction and assume $|V(\mathcal{C})| \geq 6$.

(i) By Theorem 1.4, $\mathcal{C}$ has no 3-NZF if and only if it is formed from K_4 by a series of bull-growth operations. Note that the bull-growth operation keeps the parity of degree of each vertex and each added vertex has odd degree. Thus the fact that $\mathcal{C}$ has no 3-NZF would imply that each vertex has odd degree. On the other hand, if each vertex of $\mathcal{C}$ is of odd degree, there is at least one vertex of x_1 and x_2 adjacent to u is a 3-vertex (see Fig. 3a). Then $\mathcal{C} = \mathcal{B} \uplus (\mathcal{P}(x_3, v) + x_3 v)$, where x_3 is the other common neighbor of x_1 and x_2, excepted u. Now $\mathcal{P}(x_3, v) + x_3 v$ is smaller than $\mathcal{C}$ and each vertex of it has odd degree. Thus $\mathcal{P}(x_3, v) + x_3 v$ has no 3-NZF by induction, and so $\mathcal{C}$ has no 3-NZF by Lemmas 2.5 and 2.7(ii).

(ii) Let $\psi : V(\mathcal{P}(u, v)) \to \{black, white, gray\}$ be a proper 3-coloring of $\mathcal{P}(u, v)$ with $\psi(u) = black$, and let u_1 be the first vertex of $x_1, x_2, \cdots, x_k, v$ with color *black*, assume $u_1 = x_3$. Then u_1 and u have two common neighbors and one of them has degree 3. Assume $d_{\mathcal{C}}(x_1) = 3$. (see Fig. 3b). Then $G_1 = \mathcal{C} - u - x_1 + x_3 v$ is the bull-reduction of $\mathcal{C}$ and $H = \mathcal{P}(x_3, v) + x_3 v$ is a crystal. Similar to (i), we have that either $G_1 = H$, or G_1 consists of 2-sums of H and triangles. If G_1 consists of 2-sums of H and triangles, then by

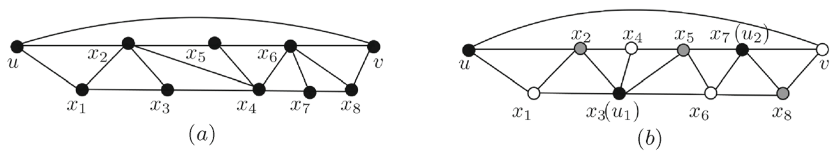

Fig. 3 The crystals in Corollary 1.2

Theorem 1.3, $G_1 \in \langle \mathbb{Z}_3 \rangle$ if and only if $H \in \langle \mathbb{Z}_3 \rangle$. $\mathcal{C} = \mathcal{B} \biguplus G_1$, by Lemma 2.6, $\mathcal{C} \in \langle \mathbb{Z}_3 \rangle$ if and only if $G_1 \in \langle \mathbb{Z}_3 \rangle$. Thus $\mathcal{C} \in \langle \mathbb{Z}_3 \rangle$ if and only if $H \in \langle \mathbb{Z}_3 \rangle$. By induction, $H = \mathcal{P}(x_3, v) + x_3 v \in \langle \mathbb{Z}_3 \rangle$ if and only if H is vertex-3-colorable, i.e., $\psi(v) \neq black$. Hence by Lemma 2.6, $\mathcal{C} \in \langle \mathbb{Z}_3 \rangle$ if and only if $\psi(v) \neq black$. Thus, (ii) holds, which completes the proof.

□

4 Two Spanning Triangle-Trees

An elementary theorem of Robbins [19] (or see Theorem 5.1 in [3]) shows that every connected graph without cut edges has a strongly connected orientation. In fact, such a strongly connected orientation can be easily obtained from ear-decompositions. This motivates the following lemma.

Lemma 4.1 *If G can be edge-partitioned into two spanning subgraphs G_1 and G_2 such that $G_1 \in \langle \mathbb{Z}_3 \rangle$ and G_2 is 2-edge-connected, then $G \in \mathcal{S}_3$.*

Proof Let β be a $\mathbb{Z}_3$-boundary of G. We first give G_2 a strongly connected orientation D_2 by Robbins' Theorem. Suppose that the boundary of G_2 corresponding to D_2 is β_2. Since $G_1 \in \langle \mathbb{Z}_3 \rangle$, there is a mod 3-orientation D_1 of G_1 for the $\mathbb{Z}_3$-boundary $\beta - \beta_2$. Since both G_1 and G_2 are spanning and D_1 is strongly connected, $D = D_1 \cup D_2$ is a strongly mod 3-orientation of G for the boundary β. That is, for any $v \in V(G)$,

$$\begin{aligned} d_D^+(v) - d_D^-(v) &= (d_{D_2}^+(v) - d_{D_2}^-(v)) + (d_{D_1}^+(v) - d_{D_1}^-(v)) \\ &\equiv \beta_2(v) + (\beta(v) - \beta_2(v)) \equiv \beta(v) (\text{mod } 3). \end{aligned}$$

So, $G \in \mathcal{S}_3$ by definition. □

Our strategy for the proof of Theorem 1.5 is to apply some extreme choice to find a 2-edge-connected spanning subgraph from one triangle-tree, and then get a $\mathbb{Z}_3$-connected spanning subgraph from another triangle-tree by adding some extra edges. We will need one more proposition before proving Theorem 1.5.

Let $\mathcal{T}$ be a triangle-tree. We say that an edge-set X of $E(\mathcal{T})$ is *removable* if $\mathcal{T} - X$ is 2-edge-connected; each edge $e \in X$ is called a removable edge.

Proposition 1 *Let $\mathcal{T}$ be a triangle-tree on $n \geq 4$ vertices with t leaves. Then $\mathcal{T}$ contains a removable set of size at least $n - t - 1$.*

Proof It is easy to check this fact for $|V(\mathcal{T})| \leq 5$. Assume it holds for $|V(\mathcal{T})| \leq k - 1$. When $|V(\mathcal{T})| = k$, let v be the new vertex added such that abv forms a new triangle. If neither a nor b is a leaf, then a largest removable set of $\mathcal{T}$ is the same as $\mathcal{T} - v$. If one of a, b is a leaf, then the edge ab is removable, and so the size of removable set increases. By induction, the proposition holds. □

Theorem 4.1 *For any graph G with $|V(G)| \geq 4$ containing two edge-disjoint spanning triangle-trees, we have $G \in \mathcal{S}_3$.*

Proof Suppose, to the contrary, that $G \notin \mathcal{S}_3$. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two edge-disjoint spanning triangle-trees of G. Let R_i be a largest removable set of $\mathcal{T}_i$ for $i = 1, 2$. Without loss of generality, assume that

$$|R_1| \geq |R_2|.$$

Our general strategy is to add the edges of R_1 to $\mathcal{T}_2$ to obtain a $\mathbb{Z}_3$-connected graph $\mathcal{T}_2 + R_1$. At the same time, $\mathcal{T}_1 - R_1$ is obviously 2-edge-connected by definition. Then it follows from Lemma 4.1 that $G \in \mathcal{S}_3$, a contradiction. Thus our ultimate goal below is to show that

$$\mathcal{T}_2 + R_1 \in \langle \mathbb{Z}_3 \rangle. \tag{4}$$

For convenience, we may also view $R_i = G[R_i]$ as an edge-induced subgraph of G. We start with the following claim.

Claim *The graph R_1 is a tree.*

Proof If R_1 contains a cycle, then by Lemma 2.4(iii) we have $\mathcal{T}_2 + R_1 \in \langle \mathbb{Z}_3 \rangle$ as desired in (4). Thus R_1 is acyclic. Let L_1 be the set of leaves in $\mathcal{T}_1$. Clearly, $L_1 \cap V(R_1) = \emptyset$ since there is no removable edge incident to a leaf. Thus by Proposition 1, we have $|R_1| \geq |V(G)| - |L_1| - 1 \geq |V(R_1)| - 1$. As R_1 is acyclic, we conclude that it is a tree. □

Claim *Let $u, w \in V(R_1)$. For any $v \in V(\mathcal{P}(u, w, \mathcal{T}_2)) \cap V(R_1)$, there is a uvw-path in R_1.*

Proof By contradiction, assume that v is not in the uw-path P_{uw} of R_1. Since R_1 is a tree by Claim 4, there is a unique shortest path from v to P_{uw} in R_1, where the intersection vertex is denoted by c. Then we have three paths P_{uc}, P_{vc}, P_{wc} intersecting at c. Note that it is possible that $c = u$ or $c = w$. Since $v \in V(\mathcal{P}(u, w, \mathcal{T}_2)) \cap V(R_1)$, we can devide $\mathcal{P}(u, w, \mathcal{T}_2)$ into two triangle-paths $\mathcal{P}(u, v, \mathcal{T}_2)$ and $\mathcal{P}(v, w, \mathcal{T}_2)$. Note that $V(\mathcal{P}(u, v, \mathcal{T}_2)) \cap V(\mathcal{P}(v, w, \mathcal{T}_2))$ contains common vertices (i.e. , u is one of their common vertex). Moreover, we have either $c \notin V(\mathcal{P}(u, v, \mathcal{T}_2))$ or $c \notin V(\mathcal{P}(v, w, \mathcal{T}_2))$. Assume, w.l.o.g., that $c \notin V(\mathcal{P}(u, v, \mathcal{T}_2))$. We lift the two paths P_{uc}, P_{vc} to become two new edges uc, vc. Then, $\mathcal{T}_2 + uc + vc \in \langle \mathbb{Z}_3 \rangle$ by Lemma 2.4 (ii), and so $\mathcal{T}_2 + P_{uc} + P_{vc} \in \langle \mathbb{Z}_3 \rangle$ by Lemmas 2.2 and 2.3. Hence, $\mathcal{T}_2 + R_1 \in \langle \mathbb{Z}_3 \rangle$, i.e., (4) holds. □

Claim *For any distinct edges $e_1 = u_1v_1 \in R_1$ and $e_2 = u_2v_2 \in R_1$, the triangle-paths $\mathcal{P}(u_1, v_1, \mathcal{T}_2)$ and $\mathcal{P}(u_2, v_2, \mathcal{T}_2)$ are edge-disjoint.*

Proof Assume it is not the case. Then $\mathcal{T}^* = \mathcal{P}(u_1, v_1, \mathcal{T}_2) \cup \mathcal{P}(u_2, v_2, \mathcal{T}_2)$ is a triangle-tree, which is a sub-triangle-tree of $\mathcal{T}_2$. Since R_1 is a tree by Claim 4, there is a shortest path connecting a vertex of e_1 and a vertex of e_2 in R_1. By appropriately relabeling the vertices, we may denote this path by $P_{u_1u_2}$. If $u_2 \in V(\mathcal{P}(u_1, v_1, \mathcal{T}_2))$, then by Claim 4 there is a $u_1u_2v_1$-path $P_{u_1u_2v_1}$ in R_1. Thus $P_{u_1u_2v_1} + u_1v_1$ is a cycle in R_1, a contradiction to Claim 4. Hence we have $u_2 \notin V(\mathcal{P}(u_1, v_1, \mathcal{T}_2))$, and so u_2 is a leaf of $\mathcal{T}^*$. Now lift the path $P_{u_1u_2}$ to become a new edge u_1u_2. Then, $\mathcal{T}^* + u_1u_2 + v_2u_2 \in \langle \mathbb{Z}_3 \rangle$ by Lemma 2.4(i). Thus, $\mathcal{T} + u_1u_2 + v_2u_2 \in \langle \mathbb{Z}_3 \rangle$ and $\mathcal{T} + R_1 \in \langle \mathbb{Z}_3 \rangle$ by Lemmas 2.2, 2.3 and Corollary 2.1. Thus, (4) holds. □

Claim *We have $|R_2| = |R_1|$, and for each $uv \in R_1$ the graph $\mathcal{P}(u, v, \mathcal{T}_2) + uv$ is a K_4.*

Proof Recall that we already have $|R_1| \geq |R_2|$ by the assumption in the beginning. It remains to show that $|R_2| \geq |R_1|$. For each edge $e = uv \in R_1$, $\mathcal{P}(u, v, \mathcal{T}_2)$ is a triangle-path with at least 4 vertices, and so it contains at least one distinguished removable edge, namely the edge in the triangle containing u but not incident to u. Moreover, all these distinguished removable edges are distinct by Claim 4 (from the fact that these triangle-paths are mutually edge-disjoint). Let R_2' be the collection of all such edges. Then, $|R_2'| \geq |R_1|$, and so by the maximality of R_2 we have $|R_2| \geq |R_2'| \geq |R_1|$. Thus, $|R_2| = |R_1|$. Furthermore, if $\mathcal{P}(u, v, \mathcal{T}_2)$ contains at least 5 vertices for some $e = uv \in R_1$, then we can easily select two removable edges from it, namely the distinguished removable edge in the triangle containing u but not incident to u and also a similar edge for v. This would result in $|R_2'| > |R_1|$, a contradiction. Hence we conclude that the graph $\mathcal{P}(u, v, \mathcal{T}_2) + uv$ is exactly a K_4 for each $uv \in R_1$.

Claim *We have $|V(G)| \geq 5$ and $|R_2| = |R_1| \geq 2$.*

Proof When $|V(G)| = 4$, it is easy to check that $G \in \mathcal{S}_3$ by Lemma 4.1. Specifically, there are three non-isomorphic distributions of $\mathcal{T}_1$ and $\mathcal{T}_2$ (see Fig. 4, we use dashed lines to distinguish $\mathcal{T}_1$ and $\mathcal{T}_2$), and G can be edge-partitioned into a spanning $\mathbb{Z}_3$-connected subgraph and a spanning 2-edge-connected subgraph in each case (thiner lines for $\mathbb{Z}_3$-connected one, broader lines for 2-edge-connected one). An alternate method for proving the case $|V(G)| = 4$ is to apply lifting techniques of Lemma 2.2 (iii), and the readers can refer to [16] for more details. Thus we have $|V(G)| \geq 5$.

Now suppose $|R_2| = |R_1| = 1$. Then both $\mathcal{T}_1$ and $\mathcal{T}_2$ contain $|V(G)| - 2$ leaves by Proposition 1. In fact, this indicates that both $\mathcal{T}_1$ and $\mathcal{T}_2$ are isomorphic to the

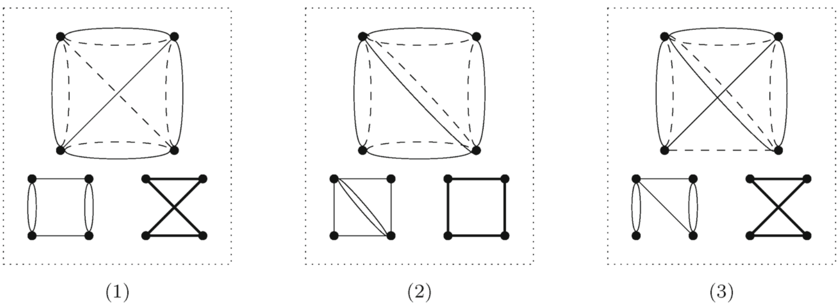

Fig. 4 Decomposition of graphs with exactly 4 vertices and 2 spanning triangle-trees

complete tripartite graph $K_{1,1,|V(G)|-2}$. As $|V(G)|\geq 5$, there are at least $|V(G)|-4\geq 1$ common leaves for $\mathcal{T}_1$ and $\mathcal{T}_2$. Let x be a common leaf of $\mathcal{T}_1$ and $\mathcal{T}_2$, and let xyz be the corresponding triangle in $\mathcal{T}_1$. Now consider the graph $G' = G - x + yz$. Then G' contains two edge-disjoint spanning triangle-trees $\mathcal{T}'_1 = \mathcal{T}_1 - x$ and $\mathcal{T}'_2 = \mathcal{T}_2 - x$. Moreover, $\mathcal{T}'_2$ is 2-edge-connected, and $\mathcal{T}'_1 + yz \in \langle \mathbb{Z}_3 \rangle$ since it contains parallel edges yz and by Corollary 2.1. Thus, $G' = G - x + yz \in \mathcal{S}_3$ by Lemma 4.1. Hence, $G \in \mathcal{S}_3$ by Lemma 2.2 (iv), a contradiction. □

The final step As in the proof of Claim 4, let R'_2 be the collection of all edges f such that $f = \mathcal{P}(u, v, \mathcal{T}_2) - u - v$ for some $uv \in R_1$. Denote $R'_2 = \{f_1, f_2, \cdots, f_s\}$, where $|R_1| = |R_2| = s$. Recall that $\mathcal{P}(f_k, f_t, \mathcal{T}_2)$ is a shortest triangle-path from f_k to f_t in $\mathcal{T}_2$. Choose $\mathcal{P}(f_k, f_t, \mathcal{T}_2)$ as small as possible among all possible distinct edges $f_k, f_t \in R'_2$.

Assume that $\mathcal{P}(f_k, f_t, \mathcal{T}_2)$ is a triangle, say uvw, where $f_k = uw$ and $f_t = vw$. We further denote the corresponding K_4 associated with f_k and f_t by $u_k u v_k w$ and $u_t u v_t w$ (see Fig. 5(1)). If $uv \in R'_2$, then R'_2 contains a cycle uvw, and so $\mathcal{T}_1 + R'_2 \in \langle \mathbb{Z}_3 \rangle$ by Lemma 2.4(iii). Thus it follows from Lemma 4.1 that $G \in \mathcal{S}_3$, a contradiction. So, we have $uv \notin R'_2$. Now let $R''_2 = R'_2 \cup \{uv\}$. Then $\mathcal{T}_2 - R''_2$ is 2-edge-connected since it contains two edge-disjoint paths $uu_k wv_t v$ and $uv_k wu_t v$ connecting u and v. Hence R''_2 is a removable set with size $|R''_2| = |R'_2| + 1 = s + 1 > s = |R_2|$, a contradiction to the maximality of R_2.

Thus $\mathcal{P}(f_k, f_t, \mathcal{T}_2)$ contains at least 4 vertices. Imaging that $\mathcal{P}(f_k, f_t, \mathcal{T}_2)$ is embedded as an outer plane graph. Let C be the outer facial cycle of $\mathcal{P}(f_k, f_t, \mathcal{T}_2)$, where $f_k, f_t \in E(C)$. Thus C is particularly a Hamiltonian cycle of $\mathcal{P}(f_k, f_t, \mathcal{T}_2)$. Then C contains a chord uv (see Fig. 5(2)). By the minimality of $\mathcal{P}(f_k, f_t, \mathcal{T}_2)$, we have $uv \notin R'_2$. Otherwise $\mathcal{P}(f_k, uv, \mathcal{T}_2)$ causes a shorter triangle-path. Now let $R''_2 = R'_2 \cup \{uv\}$. Then $\mathcal{T}_2 - R''_2$ is 2-edge-connected since u and v are still contained in a cycle similarly as aforementioned. Thus R''_2 is a removable set, which has more elements than R_2, again a contradiction. This completes the proof. □

5 Remarks on Triangularly-Connected Subgraphs

Recall the group connectivity version of Theorem 1.3 of Fan et al. [5] below.

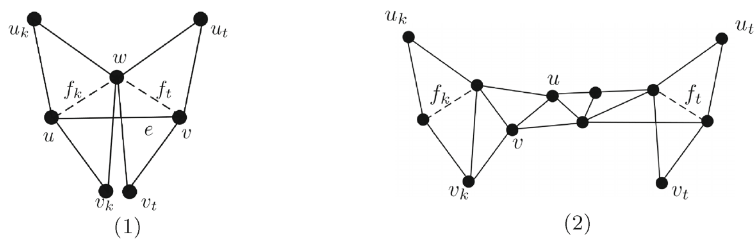

Fig. 5 The edge uv is removable in the final step in the proof of Theorem 4.1

Theorem 5.1 *Let G be a triangularly-connected graph with $|V(G)| \geq 3$. Then $G \notin \langle \mathbb{Z}_3 \rangle$ if and only if there is a subgraph G_1 and an odd wheel or a triangle, called W, such that $G = W \bigoplus_2 G_1$, where $G_1 \notin \langle \mathbb{Z}_3 \rangle$ is triangularly-connected.*

From this theorem, we can easily characterize triangularly-connected graphs without spanning triangle-trees under the assumption of $\mathbb{Z}_3$-connectivity. An *eccentrical edge* of a wheel is an edge that is not incident with the center vertex. A wheel in a graph G is *fully 2-summed* if for each eccentrical edge e, there exist subgraphs A, B of G such that $G = A \bigoplus_2 B$ and $E(A) \cap E(B) = \{e\}$ (see Fig. 6 below).

Proposition 2 *Let $G \notin \langle \mathbb{Z}_3 \rangle$ be a triangularly-connected graph. Then G has no spanning triangle-tree if and only if there is an odd wheel of G that is fully 2-summed.*

Proof The "if" part is trivial, since each eccentrical edge of the fully 2-summed odd wheel must be in the spanning triangle-tree, which leads to a contradiction. It remains to justify the "only if" part.

Suppose, to the contrary, that $\mathcal{T}$ is a maximum triangle-tree of G, where $|V(\mathcal{T})| < |V(G)|$. Then there exists a pair of incident edges e_1,e_2 with $e_1 \in E(\mathcal{T})$, $e_2 \notin E(\mathcal{T})$, where e_1 and e_2 are intersecting at $v \in V(\mathcal{T})$. Since G is triangularly-connected, there is a triangle-path $\mathcal{P}$ from e_1 to e_2. So, there must be a triangle with 2 vertices in $V(\mathcal{T})$, named x, y, and one vertex in $V(G) - V(\mathcal{T})$, named z. If $xy \in E(\mathcal{T})$, then $\mathcal{T} + xz + yz$ is a larger triangle-tree, a contradiction. So, we have $xy \notin E(\mathcal{T})$ and there is a triangle xyt on $\mathcal{P}$ with $t \in V(T)$. If there is at most one edge of xt, yt in $E(\mathcal{T})$, say yt, then by Lemma 2.2(i), $T + xy + xt \in \langle \mathbb{Z}_3 \rangle$. Thus, $G \in \langle \mathbb{Z}_3 \rangle$ by Lemma 2.1(iii). So, both xt and yt are in $E(\mathcal{T})$. Since $\mathcal{T}$ is a triangle-tree, there is a triangle-path $\mathcal{Q}$ from xt to yt. Moreover, $\mathcal{Q}$ is a fan, a wheel with one eccentrical edge deleted. If there is an eccentrical edge f not in any 2-sum in $G - \mathcal{Q}$, then $\mathcal{T} - f + xy + xz + yz$ is a larger triangle-tree of G, a contradiction. So, G has a fully 2-summed wheel. The proof is thus complete. □

From Theorem 5.1 and Proposition 2, non-$\mathbb{Z}_3$-connected triangularly-connected graphs almost have the same structure as graphs containing spanning triangle-trees. Each of them is formed from some well-characterized building blocks (triangles and odd wheels) by applying some 2-sum operations. Thus all the main results concerning spanning triangle-trees in this paper can be easily transferred to graphs

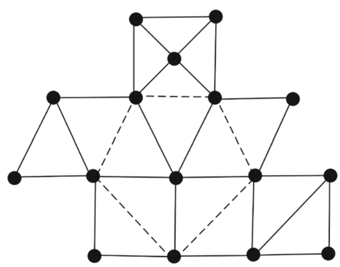

Fig. 6 A wheel that is fully 2-summed

containing spanning triangularly-connected subgraphs, with essentially the same proof. For example, we have the following more general theorem.

Theorem 5.2 *Let G be a graph containing a spanning triangularly-connected subgraph.*

(a) G has no 3-NZF if and only if $G = \mathcal{B} \uplus G_1$, where G_1 contains a spanning triangularly-connected subgraph and has no 3-NZF. In other words, G has no 3-NZF if and only if G is formed from K_4 by a series of bull-growth operations.
(b) $G \notin \langle \mathbb{Z}_3 \rangle$ if and only if G can be constructed from K_3 or K_4 by 2-sum and bull-growth operations.

The methods developed in this paper may be helpful in studying the following more general problem.

Problem 1 *Let $\mathcal{F}$ be the family of all graphs G such that for any $u, v \in V(G)$ there is uv-triangle-path in G. Characterize all graphs in $\mathcal{F}$ that admits a 3-NZF.*

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