Discrete Mathematics 343 (2020) 111809

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Complementary graphs with flows less than three

Jiaao Li [a,*], Xueliang Li [b], Meiling Wang [b]

[a] School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China
[b] Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, China

ARTICLE INFO

Article history:
Received 30 July 2019
Received in revised form 23 December 2019
Accepted 29 December 2019
Available online 9 January 2020

Keywords:
Nowhere-zero flow
Flow index
Strongly connected orientation
Contractible configuration
Complementary graphs

ABSTRACT

We prove that for a simple graph G with $|V(G)| \geq 32$, if $\min\{\delta(G), \delta(G^c)\} \geq 4$, then either G or its complementary graph G^c has flow index strictly less than 3. This is proved by a newly developed closure operation, which may be useful in studying further flow index problems. In particular, our result supports a recent conjecture of Li et al. (2018), and improves a result of Hou et al. (2012) on nowhere-zero 3-flows.

1. Introduction

Graphs in this paper may contain parallel edges but no loops. We call a graph *simple* if it contains no parallel edges. An integer flow of a graph G is an ordered pair (D, f), where D is an orientation of G and f is a mapping from $E(G)$ to the set of integers such that the incoming netflow equals the outgoing netflow at every vertex. A flow (D, f) is called a *nowhere-zero k-flow* if $f(e) \in \{\pm 1, \pm 2, \ldots, \pm(k-1)\}$ for every edge $e \in E(G)$. Tutte proposed several celebrated flow conjectures, and the 3-flow conjecture is stated as follows.

Conjecture 1.1 (*Tutte's 3-Flow Conjecture, 1972*)**.** *Every 4-edge-connected graph has a nowhere-zero 3-flow.*

Jaeger [3] in 1979 showed that every 4-edge-connected graph has a nowhere-zero 4-flow. In 2012, Thomassen [9] made a breakthrough on this conjecture by showing that every 8-edge-connected graph has a nowhere-zero 3-flow. This was later improved by Lovász, Thomassen, Wu and Zhang [6].

Theorem 1.2 (*Lovász et al. [6]*)**.** *Every 6-edge-connected graph has a nowhere-zero 3-flow.*

Besides the edge connectivity conditions, Hou, Lai, Li and Zhang [2] studied the 3-flow property of a graph G and its complementary graph G^c, providing another evidence to Tutte's 3-flow conjecture.

Theorem 1.3 (*Hou et al. [2]*)**.** *Let G be a simple graph with $|V(G)| \geq 44$. If $\min\{\delta(G), \delta(G^c)\} \geq 4$, then either G or G^c has a nowhere-zero 3-flow.*

* Corresponding author.
E-mail addresses: lijiaao@nankai.edu.cn (J. Li), lxl@nankai.edu.cn (X. Li), Estellewml@gmail.com (M. Wang).

https://doi.org/10.1016/j.disc.2019.111809

For integers $k \geq 2d > 0$, a *circular k/d-flow* is an integer flow (D, f) such that f takes values from $\{\pm d, \pm(d+1), \ldots, \pm(k-d)\}$. When $d = 1$, this is exactly the nowhere-zero k-flow. The *flow index* $\phi(G)$ of a graph G is the least rational number r such that G admits a circular r-flow. It was proved in [1] that such an index indeed exists, and the circular flow satisfies the monotonicity that for any pair of rational numbers $r \geq s$, a graph admitting a circular s-flow has a circular r-flow as well. Thus circular flows are refinements of integer flows.

A *modulo 3-orientation* is an orientation D of G such that the outdegree is congruent to the indegree modulo 3 at each vertex. It is well-known that a graph admits a nowhere-zero 3-flow if and only if it admits a modulo 3-orientation (see [4,11,12]). An orientation is *strongly connected* if for any two vertices $x, y \in V(G)$, there is a directed path from x to y. The study of flow index strictly less than 3 was initiated in [5] with the following theorems.

Theorem 1.4 ([5]). *A connected graph G satisfies $\phi(G) < 3$ if and only if G has a strongly connected modulo 3-orientation.*

Theorem 1.5 ([5]). *For every 8-edge-connected graph G, the flow index $\phi(G) < 3$.*

It is worth noting that (see [4]) if $\phi(G) \leq 5/2$ for every 9-edge-connected graph G, then Tutte's 5-Flow Conjecture follows, that is, $\phi(G) \leq 5$ for every bridgeless graph G. Since K_6 has only one modulo 3-orientation up to isomorphism which is not strongly connected, we have $\phi(K_6) = 3$, and so Theorem 1.5 cannot be extended to 5-edge-connected graphs. In [5], it was conjectured that the 6-edge-connectivity suffices for $\phi(G) < 3$.

Conjecture 1.6 ([5]). *For every 6-edge-connected graph G, the flow index $\phi(G) < 3$.*

In this paper, we aim to extend Theorem 1.3 in the theme of flow index $\phi < 3$. Our main result is as follows, providing further evidence to Conjecture 1.6.

Theorem 1.7. *Let G be a simple graph with $|V(G)| \geq 32$. If $\min\{\delta(G), \delta(G^c)\} \geq 4$, then $\min\{\phi(G), \phi(G^c)\} < 3$.*

Theorems 1.2 and 1.3 were proved by using the *group connectivity* ideas, which allows flow with boundaries. Let G be a graph, and let $Z(G, \mathbb{Z}_3) = \{\beta : V(G) \to \mathbb{Z}_3 \mid \sum_{v \in V(G)} \beta(v) \equiv 0 \pmod 3\}$. Given a boundary function $\beta \in Z(G, \mathbb{Z}_3)$, an orientation D of G is called a *β-orientation* if $d_D^+(v) - d_D^-(v) \equiv \beta(v) \pmod 3$ for every vertex $v \in V(G)$. A graph G is *$\mathbb{Z}_3$-connected* if G has a β-orientation for every $\beta \in Z(G, \mathbb{Z}_3)$. It follows from the definition that every $\mathbb{Z}_3$-connected graph admits a modulo 3-orientation and hence has a nowhere-zero 3-flow. In fact, Hou et al. [2] obtained a stronger version of Theorem 1.3 on $\mathbb{Z}_3$-group connectivity.

Theorem 1.8 (*Hou et al. [2]*). *Let G be a simple graph with $|V(G)| \geq 44$. If $\min\{\delta(G), \delta(G^c)\} \geq 4$, then either G or G^c is $\mathbb{Z}_3$-connected.*

Motivated by Theorem 1.4, we develop a contractible configuration method to handle the flow index $\phi < 3$ problem in this paper, which is analogous to the $\mathbb{Z}_3$-group connectivity.

Definition 1.9. A graph G is **strongly connected $\mathbb{Z}_3$-contractible** if, for every $\beta \in Z(G, \mathbb{Z}_3)$, there is a strongly connected orientation D such that $d_D^+(v) - d_D^-(v) \equiv \beta(v) \pmod 3$ for every vertex $v \in V(G)$. Let $\mathcal{S}_3$ denote the family of all strongly connected $\mathbb{Z}_3$-contractible graphs.

A strongly connected $\mathbb{Z}_3$-contractible graph is called an **$\mathcal{S}_3$-graph** for convenience. An $\mathcal{S}_3$-graph is $\mathbb{Z}_3$-connected by definition; and it has flow index less than 3 by Theorem 1.4. Actually, it was proved in Theorem 4.2 of [5] that $G \in \mathcal{S}_3$ for every 8-edge-connected graph G.

In this paper, we shall prove an $\mathcal{S}_3$ version of Theorem 1.7. However, a directed $\mathcal{S}_3$-property like Theorem 1.8 fails, and there are some exceptions. A **bad attachment** of a graph G is an induced subgraph Γ with $3 \leq |V(\Gamma)| \leq 6$ and there are at most $3|V(\Gamma)| - |E(\Gamma)|$ edges between $V(\Gamma)$ and $V(G) \setminus V(\Gamma)$ in G. We will see later (Remark 2 in Section 2) that if a graph G contains a bad attachment, then $G \notin \mathcal{S}_3$. We obtain the $\mathcal{S}_3$ version of Theorem 1.7 as follows.

Theorem 1.10. *Let G be a simple graph with $|V(G)| \geq 77$. If $\min\{\delta(G), \delta(G^c)\} \geq 4$, then one of the following holds:*

(i) $G \in \mathcal{S}_3$ or $G^c \in \mathcal{S}_3$,
(ii) both G and G^c contain a bad attachment.

Moreover, in case (ii) we have both $\phi(G) < 3$ and $\phi(G^c) < 3$.

In fact, if case (ii) of Theorem 1.10 occurs, we obtain a more detailed characterization of bad attachments in Theorem 3.3 of Section 3. Also, the graph obtained by deleting bad attachment(s) is a special kind of contractible graph for $\phi < 3$ property, to be introduced in Section 2. Furthermore, if we impose the minimal degree condition to $\min\{\delta(G), \delta(G^c)\} \geq 5$, then an easy counting argument shows that case (ii) of Theorem 1.10 cannot happen; see Theorem 3.3 for more details. Thus we have the following corollary.

Corollary 1.11. *Let G be a simple graph with $|V(G)| \geq 77$. If $\min\{\delta(G), \delta(G^c)\} \geq 5$, then $G \in \mathcal{S}_3$ or $G^c \in \mathcal{S}_3$.*

In the next section, we will present some preliminaries. The proofs of Theorems 1.7 and 1.10 will be given in Section 3. We end this section with a few more notation.

Notation. A vertex of degree at least k is called a k^+-*vertex*. Let X, Y be two disjoint subsets of vertices in a graph G. We denote the set of edges between X and Y in G by $E_G(X, Y)$, and let $e_G(X, Y) = |E_G(X, Y)|$. When $X = \{x\}$ or $Y = \{y\}$, we use $E_G(x, Y)$, $E_G(X, y)$, $E_G(x, y)$ and $E_G(u) = E_G(\{u\}, V(G) \setminus \{u\})$ for short. For a vertex set $A \subseteq V(G)$, we denote by G/A the graph obtained from G by identifying the vertices of A into a single vertex and deleting the resulting loops. For an edge set $B \subseteq E(G)$, denote by G/B the graph obtained from G by identifying the endpoints of each edge one by one and deleting the resulting loops. Moreover, we use G/H for $G/V(H)$ when H is a connected subgraph of G.

2. Preliminaries

The following observation comes straightly from Definition 1.9 of an $\mathcal{S}_3$-graph. This indicates that the $\mathcal{S}_3$-property is closed under contraction of vertices and addition of edges. It would also be useful for determining some graphs not in $\mathcal{S}_3$.

Observation 2.1. *Let x, y be two vertices of G. If $G \in \mathcal{S}_3$, then $G + xy \in \mathcal{S}_3$ and $G/\{x, y\} \in \mathcal{S}_3$. Conversely, if there is a subset $X \subsetneq V(G)$ of vertices such that $G/X \notin \mathcal{S}_3$, then $G \notin \mathcal{S}_3$.*

2.1. Contractible configurations and 3-closure operations

Lemma 2.2. *Let G be a connected graph with $\beta \in Z(G, \mathbb{Z}_3)$, and H a connected subgraph of G and $G' = G/H$. Define a boundary function β' of G' as follows.*

$$\beta'(v) = \begin{cases} \beta(v), & \text{if } v \in V(G/H) \setminus \{v_H\}, \\ \displaystyle\sum_{x \in V(H)} \beta(x), & \text{if } v = v_H, \end{cases}$$

where v_H denotes the vertex by contracting H in G'. Then $\beta' \in Z(G', \mathbb{Z}_3)$.

If $H \in \mathcal{S}_3$, then every strongly connected β'-orientation of G' can be extended to a strongly connected β-orientation of G. In particular, each of the following statements holds.

(i) If $H \in \mathcal{S}_3$ and $\phi(G/H) < 3$, then $\phi(G) < 3$.
(ii) If $H \in \mathcal{S}_3$ and $G/H \in \mathcal{S}_3$, then $G \in \mathcal{S}_3$.

Proof. Since $\sum_{x \in V(G')} \beta'(x) = \sum_{x \in V(G) \setminus V(H)} + \sum_{x \in V(H)} \beta(x) \equiv 0 \pmod 3$, we have $\beta' \in Z(G', \mathbb{Z}_3)$. For a strongly connected β'-orientation D' of G', it results a β_1-orientation D_1 of $G - E(H)$ (we may arbitrarily orient the edges in $E(G[V(H)]) \setminus E(H)$ here). Define a function $\beta_2 : V(H) \mapsto \mathbb{Z}_3$ by $\beta_2(v) = \beta(v) - \beta_1(v)$ for each $v \in V(H)$. Then $\sum_{v \in V(H)} \beta_2(v) = \sum_{v \in V(H)} \beta(v) - \sum_{v \in V(H)} \beta_1(v) = \beta'(v_H) - (d^+_{D'}(v_H) - d^-_{D'}(v_H)) \equiv 0 \pmod 3$, and so $\beta_2 \in Z(H, \mathbb{Z}_3)$. Since $H \in \mathcal{S}_3$, there is a strongly connected β_2-orientation D_2 of H. Now $D_1 \cup D_2$ is a β-orientation of G. Since both D_2 and $D' = (D_1 \cup D_2)/D_2$ are strongly connected, $D_1 \cup D_2$ is strongly connected.

(i) If $H \in \mathcal{S}_3$, then a strongly connected modulo 3-orientation of G/H can be extended to G. Hence (i) follows from Theorem 1.4.

(ii) Since β is arbitrary, $G \in \mathcal{S}_3$ by definition. ■

Since a graph with 3-edge-cuts cannot have a strongly connected modulo 3-orientation, it has flow index at least 3 by Theorem 1.4. So our study of flow index $\phi < 3$ only focuses on 4-edge-connected graphs. A graph H is called *($\phi < 3$)-contractible* if for every 4-edge-connected supergraph G containing H as a subgraph, $\phi(G) < 3$ if and only if $\phi(G/H) < 3$. Clearly, an $\mathcal{S}_3$-graph is ($\phi < 3$)-contractible by (i) of Lemma 2.2. We will show below that a wider class of graphs is also ($\phi < 3$)-contractible.

Lemma 2.3 ([5])**.** *Let G be a 2-edge-connected graph, and $e = xy$ an edge of G. If G/e has a strongly connected orientation D', then D' can be extended to a strongly connected orientation D of G.*

Lemma 2.4. *Let G be a 4-edge-connected graph with $\beta \in Z(G, \mathbb{Z}_3)$ and x, y be a pair of vertices joined by a set $E(x, y)$ of at least 3 parallel edges. Let $G' = G/E(x, y)$ and β' be the resulting $\mathbb{Z}_3$ boundary function, where $\beta'(v) = \beta(v)$ for any $v \in V(G) \setminus \{x, y\}$, and $\beta'(w) \equiv \beta(x) + \beta(y) \pmod 3$ for the contracted vertex w. If G' has a strongly connected β'-orientation D', then D' can be extended to a strongly connected β-orientation D of G.*

Proof. Let e_1, e_2 be two distinct parallel edges in $E(x, y)$. Then $G - e_1 - e_2$ is 2-edge-connected since G is 4-edge-connected, and hence we can extend D' to a strongly connected orientation of $G - e_1 - e_2$ by Lemma 2.3. Note that two parallel edges e_1, e_2 are enough to modify the boundaries of the end vertices x, y. Now we appropriately orient e_1, e_2 to modify the boundary $\beta(x)$, $\beta(y)$. This results in a strongly connected β-orientation D of G. ■

Remark 1. For a vertex $v \in V(G)$ and a boundary $\beta \in Z(G, \mathbb{Z}_3)$, two edges incident to v are enough to *modify* $\beta(v)$. Specially, when $\beta(v) = 0$, orient the two edges oppositely; when $\beta(v) = 1$, orient both edges towards v; when $\beta(v) = 2$, orient both edges away from v. If k edges are incident to v (where $k \geq 2$), we can first orient $k-2$ edges arbitrarily, and then orient the remaining two edges to achieve $\beta(v)$. This fact will be frequently used in this paper implicitly.

In particular, Lemma 2.4 indicates that the graph formed by three or more parallel edges is $(\phi < 3)$-contractible.

Definition 2.5. Let H be a connected subgraph of G. The **3-closure** of H in G, denoted by $cl_3(H)$, is the unique maximal induced subgraph of G that contains H such that $V(cl_3(H)) \setminus V(H)$ can be ordered as a sequence $\{v_1, v_2, \ldots, v_t\}$ such that $e_G(v_1, V(H)) \geq 3$ and for each i with $1 \leq i \leq t-1$,

$$e_G(v_{i+1}, V(H) \cup \{v_1, v_2, \ldots, v_i\}) \geq 3.$$

Notice that for each vertex $v \in V(G) \setminus V(cl_3(H))$, we have $e_G(v, cl_3(H)) \leq 2$ by the definition. The following lemma tells that if $H \in \mathcal{S}_3$, then $cl_3(H)$ is also $(\phi < 3)$-contractible.

Lemma 2.6. *Let G be a 4-edge-connected graph with a subgraph H. Then each of the following statements holds.*

(i) If $H \in \mathcal{S}_3$ and $\phi(G/cl_3(H)) < 3$, then $\phi(G) < 3$.
(ii) If $H \in \mathcal{S}_3$ and $G/cl_3(H) \in \mathcal{S}_3$, then $G \in \mathcal{S}_3$.

Proof. (i) Let $\{v_1, v_2, \ldots, v_t\}$ be the ordered sequence of $V(cl_3(H)) \setminus V(H)$ as in Definition 2.5. Denote $H_i = G[V(H) \cup \{v_1, v_2, \ldots, v_{t+1-i}\}]$ for each $1 \leq i \leq t$ and $H_{t+1} = H$. By Lemma 2.4, we first extend a strongly connected modulo 3-orientation of $G/cl_3(H) = G/H_1$ to G/H_2. By applying Lemma 2.4 recursively, we can extend a strongly connected modulo 3-orientation of G/H_i to G/H_{i+1} for each $i = 1, 2, \ldots, t$. Then we apply Lemma 2.2 to extend this strongly connected modulo 3-orientation of G/H to a strongly connected modulo 3-orientation of G.

(ii) The proof of (ii) is similar to that of (i) with strongly connected β-orientation replacing strongly connected modulo 3-orientation. ∎

2.2. Properties of contractible graphs

By Theorem 4.2 of [5], we have the following theorem.

Theorem 2.7 ([5]). *For every 8-edge-connected graph G, $G \in \mathcal{S}_3$.*

A graph is called *trivial* if it is a singleton K_1, and *nontrivial* otherwise. The following lemma is due to Nash-Williams [8] in terms of matroids, and a detailed proof can be found in Theorem 2.4 of [10].

Lemma 2.8 (Nash-Williams [8]). *Let G be a nontrivial graph and let $k > 0$ be an integer. If $|E(G)| \geq k(|V(G)| - 1)$, then G has a nontrivial subgraph H such that H contains k edge-disjoint spanning trees.*

Theorem 2.7 and Lemma 2.8 immediately imply the following lemma, which shows that graphs with enough edges must have a nontrivial $\mathcal{S}_3$-subgraph.

Lemma 2.9. *Let G be a simple graph with $|E(G)| \geq 8(|V(G)| - 1)$. Then G has a nontrivial subgraph $H \in \mathcal{S}_3$ with $|V(H)| \geq 16$.*

Proof. By Lemma 2.8, G has a nontrivial subgraph H that contains 8 edge-disjoint spanning trees. Clearly, H is 8-edge-connected, and so $H \in \mathcal{S}_3$ by Theorem 2.7. If H is a simple graph, then $|V(H)| \geq 16$ follows from that H contains 8 edge-disjoint spanning trees. ∎

On the other hand, we also show that an $\mathcal{S}_3$-graph cannot be too sparse.

Lemma 2.10. *If a nontrivial graph G belongs to $\mathcal{S}_3$, then $|E(G)| \geq 3|V(G)| - 2$.*

Proof. Fix a vertex $x \in V(G)$, define a boundary function $\beta : V(G) \to \mathbb{Z}_3$ by

$$\beta(v) \equiv \begin{cases} \displaystyle\sum_{y \in V(G)\setminus\{x\}} d_G(y) \pmod 3, & \text{if } v = x, \\ -d_G(v) \pmod 3, & \text{if } v \neq x. \end{cases}$$

Clearly, $\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod 3$ and $\beta \in Z(G, \mathbb{Z}_3)$. Since $G \in \mathcal{S}_3$, there is a strongly connected β-orientation D of G, that is, $\beta(v) \equiv d_D^+(v) - d_D^-(v) = 2d_D^+(v) - d_G(v) \pmod 3$ for any vertex $v \in V(G)$. For any vertex $v \in V(G) \setminus \{x\}$, since

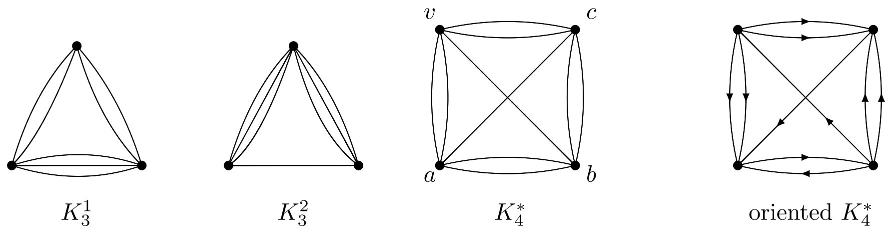

Fig. 1. The graphs K_3^1, K_3^2, K_4^* and strongly connected mod 3-orientation of K_4^*.

$\beta(v) \equiv -d_G(v) \pmod 3$, we have $d_D^+(v) \equiv 0 \pmod 3$, and so $d_D^+(v) \geq 3$ as a positive integer since D is strongly connected. Moreover, $d_D^+(x) \geq 1$ since D is strongly connected. Therefore,

$$|E(G)| = \sum_{v \in V(G)} d_D^+(v) = d_D^+(x) + \sum_{v \in V(G)\setminus\{x\}} d_D^+(v) \geq 1 + 3(|V(G)| - 1) = 3|V(G)| - 2. \quad \blacksquare$$

Remark 2. If a graph G contains a bad attachment Γ, then for $X = V(G) \setminus V(\Gamma)$, the graph G/X has $|V(\Gamma)| + 1$ vertices and at most $3|V(\Gamma)|$ edges. Thus $G/X \notin \mathcal{S}_3$ by Lemma 2.10, and so $G \notin \mathcal{S}_3$ by Observation 2.1.

Now we develop some techniques to find $\mathcal{S}_3$-graphs from smaller graphs. For a graph G with a 4^+-vertex v and $va, vb \in E_G(v)$, define $G_{[v,ab]} = G - v + ab$ as the graph obtained from G by deleting the vertex v and adding a new edge ab. We refer this operation as *splitting* edges va, vb to become a new edge ab.

Lemma 2.11. *Let v be a 4^+-vertex of a graph G with $va, vb \in E_G(v)$. If $G_{[v,ab]} \in \mathcal{S}_3$, then $G \in \mathcal{S}_3$.*

Proof. Let $\beta \in Z(G, \mathbb{Z}_3)$. We first orient all the edges of $E_G(v) \setminus \{va, vb\}$ to modify the boundary $\beta(v)$. Note that this is possible since $|E_G(v) \setminus \{va, vb\}| \geq 2$. Then delete the oriented edges and change the boundaries of the end vertices other than v. Specifically, for each edge $vx \in E_G(v) \setminus \{va, vb\}$ that we oriented, increase or decrease the boundary function of x by 1 depending on the orientation of vx that is into x or out of x. This results in a boundary function β' of $G_{[v,ab]}$. Since $G_{[v,ab]} \in \mathcal{S}_3$, there exists a strongly connected β'-orientation D' of $G_{[v,ab]}$. By adding those deleted oriented edges and replacing the edge ab by av, vb with their orientations the same as ab (if $a = b$, orient av, vb as a directed 2-cycle), we obtain a strongly connected β-orientation of G. This argument holds for any $\beta \in Z(G, \mathbb{Z}_3)$, and hence $G \in \mathcal{S}_3$. $\blacksquare$

Lemma 2.12. *Let G be a 4-edge-connected graph and u, v be two adjacent vertices in G. Assume that $e_G(v, V(G) \setminus \{u, v\}) \geq 3$ and let $va, vb \in E(v, V(G) \setminus \{u, v\})$. Denote $G_1 = G - u - v + ab$. If $G_1 \in \mathcal{S}_3$, then $G \in \mathcal{S}_3$.*

Proof. If u has just one neighbor v, then there are at least 4 parallel edges between u and v. We denote the graph G/uv by H. Since $H_{[v,ab]} = G_1 \in \mathcal{S}_3$ and by Lemma 2.11, $H \in \mathcal{S}_3$. Hence $G \in \mathcal{S}_3$ by Lemmas 2.4 and 2.6.

So we assume that u has at least two neighbors. Let $c \neq v$ be a neighbor of u, and $H = G - u + vc$. Then $H_{[v,ab]} = G - u - v + ab = G_1 \in \mathcal{S}_3$. Since $e_G(v, V(G) \setminus \{u, v\}) \geq 3$, we know that v is a 4^+-vertex of H, and so $H \in \mathcal{S}_3$ by Lemma 2.11. Notice that u is a 4^+-vertex of G and $H = G_{[u,vc]} \in \mathcal{S}_3$. Hence $G \in \mathcal{S}_3$ by Lemma 2.11 again. $\blacksquare$

Remark 3. The condition "$e_G(v, V(G) \setminus \{u, v\}) \geq 3$" in Lemma 2.12 cannot be dropped. If there are exactly two parallel edges between u and v in G and both u and v have exactly two other edges connecting $V(G) \setminus \{u, v\}$, then this graph G does not belong to $\mathcal{S}_3$ by Observation 2.1.

2.3. Special contractible graphs

Let mK_2 be the graph with two vertices and m parallel edges. Let K_3^1, K_3^2, and K_4^* be the graphs as depicted in Fig. 1.

Lemma 2.13. *(i) $mK_2 \in \mathcal{S}_3$ if and only if $m \geq 4$.*
(ii) $K_3^1, K_3^2, K_4^* \in \mathcal{S}_3$.

Proof. (i) By Lemma 2.10, we have that $mK_2 \in \mathcal{S}_3$ implies $m \geq 4$. When $m \geq 4$, we first orient two of the edges in the opposite directions to obtain a digon. Then there are at least two edges remaining, and we can use them to modify the boundaries of end vertices. This gives a strongly connected β-orientation for any given boundary function β, and so $mK_2 \in \mathcal{S}_3$.

(ii) For K_3^1, K_3^2, each of them contains a $3K_2$, and contracting a $3K_2$ results a $4K_2 \in \mathcal{S}_3$. So $K_3^1, K_3^2 \in \mathcal{S}_3$ by Lemma 2.4.

Let $\beta \in Z(K_4^*, \mathbb{Z}_3)$. If $\beta = 0$ at each vertex, then a strongly connected modulo 3-orientation of K_4^* is in the last graph of Fig. 1. Otherwise, without loss of generality, we may assume $\beta(v) = \alpha \in \{-1, 1\}$. Consider a graph $G_1 = K_4^* - v + ab + ac$ with boundary β_1 such that $\beta_1(a) = \beta(a)$, $\beta_1(b) = \beta(b)$ and $\beta_1(c) = \beta(c) + \alpha$. Then $\beta_1 \in Z(G_1, \mathbb{Z}_3)$ and $G_1 \cong K_3^1 \in \mathcal{S}_3$, and there exists a strongly connected β_1-orientation of G_1. In K_4^*, replace the added edges ab, ac by av, vb and av, vc with their orientations preserved, respectively. Then orient the remaining edge vc of K_4^* from v to c if $\alpha = 1$, and from c to v if $\alpha = -1$. This gives a strongly connected β-orientation of K_4^*. Hence $K_4^* \in \mathcal{S}_3$. ■

Now we show that some complete bipartite graphs are in $\mathcal{S}_3$. Note that $K_{4,9}$ has 13 vertices and 36 edges, and so $K_{4,9} \notin \mathcal{S}_3$ by Lemma 2.10.

Lemma 2.14. *When $m \geq 4$ and $n \geq 10$, we have $K_{m,n} \in \mathcal{S}_3$.*

Proof. We first show $K_{4,10} \in \mathcal{S}_3$. Let (X, Y) be a bipartition of $K_{4,10}$ with $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_i | 1 \leq i \leq 10\}$. We apply Lemma 2.11 to delete vertices in Y and add edges in X. For $1 \leq i \leq 4$, we delete y_{2i-1}, y_{2i} and add two parallel edges $x_i x_{i+1}$, where $x_5 = x_1$. Then delete y_9, y_{10} and add edges x_1x_3, x_2x_4. Now the remaining graph is isomorphic to $K_4^* \in \mathcal{S}_3$. By applying Lemma 2.11 recursively, we conclude that $K_{4,10} \in \mathcal{S}_3$.

When $m \geq 4$ and $n \geq 10$, $K_{m,n}$ is 4-edge-connected. Pick a subgraph $K_{4,10}$ in $K_{m,n}$. Then it is easy to see that $K_{m,n} = cl_3(K_{4,10})$. Since $K_{4,10} \in \mathcal{S}_3$, we have $K_{m,n} \in \mathcal{S}_3$ by Lemma 2.6(ii). ■

By Observation 2.1, if a graph G contains $K_{m,n} \in \mathcal{S}_3$ as a spanning subgraph with $m \geq 4$ and $n \geq 10$, then $G \in \mathcal{S}_3$. We shall prove a similar proposition below when G contains $K_{3,t}$ as a spanning subgraph and t is large ($t \geq 14$ suffices).

For an integer $t \geq 4$, a 4-edge-connected graph on $t + 3$ vertices is denoted by $K_{3,t}^+$ if it contains $K_{3,t}$ as a spanning subgraph.

Lemma 2.15. *For $t \geq 14$, $K_{3,t}^+ \in \mathcal{S}_3$.*

Proof. Let (A, B) be the bipartition of $G = K_{3,t}^+$ with $|A| = t$, $|B| = 3$ and $E(A, B)$ contains a complete bipartite graph $K_{3,t}$. Denote $B = \{x, y, z\}$. Our strategy is to apply Lemmas 2.11 and 2.12 to delete vertices in A and add edges to B such that part B forms a graph $K_3^1 \in \mathcal{S}_3$. Note that in part B, we need to add at most 7 edges to form a K_3^1. In part A, we can delete a vertex or two adjacent vertices and add any one of xy, xz, yz by using Lemmas 2.11 and 2.12. We will proceed to add two parallel edges xy, two parallel edges xz and three parallel edges yz. The only concern is that we need to keep the remaining graph 4-edge-connected.

Let $C_1, C_2, \ldots, C_s$ be all the components of $G[A]$. Given a component C_i where $1 \leq i \leq s$. We first note that the operations of the following cases keep the remaining graph 4-edge-connected. If $|V(C_i)| = 1$, then it means that there are parallel edges between $V(C_i)$ and some vertex of B, and we can delete the vertex $V(C_i)$ and add a new edge in B by using Lemma 2.11. If $|V(C_i)| = 2$, then there are two adjacent vertices u, v in $V(C_i)$. Clearly, $e_G(v, B) \geq 3$ and Lemma 2.12 is applied. In this case we delete $V(C_i)$ and add a new edge in B. If $|V(C_i)| \geq 3$, we pick a spanning tree of C_i, and then delete a pendent vertex in the tree and add a new edge in B by using Lemma 2.11 iteratively, until this component becomes two adjacent vertices. Now we use Lemma 2.12 to delete this last two vertices and add a new edge in B. In total, all those operations could add at least

$$\sum_{|V(C_i)| \geq 2} (|V(C_i)| - 1) + \sum_{|V(C_i)|=1} |V(C_i)| \geq \sum_{|V(C_i)| \geq 2} \frac{|V(C_i)|}{2} + \sum_{|V(C_i)|=1} |V(C_i)| \geq \frac{1}{2} \sum_{i=1}^{s} |V(C_i)| \geq 7$$

edges to part B.

Therefore, we can successfully apply these operations to obtain a $K_3^1 \in \mathcal{S}_3$ in part B, and the resulting graph is 4-edge-connected and it is formed by $cl_3(K_3^1)$. Hence it is in $\mathcal{S}_3$ by (ii) of Lemma 2.6. By using Lemmas 2.11 and 2.12 recursively, we can get $K_{3,t}^+ \in \mathcal{S}_3$. ■

As mentioned in the introduction, we have $\phi(K_6) = 3$; and there is another 5-edge-connected planar graph $2C_5 \cdot K_1$ on 6 vertices as in Fig. 2 with flow index exactly 3 (see Section 5 in [5]). We shall show below that 4-edge-connected graphs with fewer vertices have flow index less than 3.

Lemma 2.16. *For a 4-edge-connected graph G on $n \leq 5$ vertices, $\phi(G) < 3$.*

Proof. When $n \leq 2$, it holds by (i) of Lemma 2.13. Suppose that G is a minimal counterexample of the lemma with the least vertices. Then $|V(G)| \geq 3$ and G has no strongly connected modulo 3-orientation.

Mader's splitting lemma [7] tells that, if G has a vertex v of even degree, then we can split all the edges incident to v in pairs such that the resulting graph H remains 4-edge-connected. Clearly, a strongly connected modulo 3-orientation of H extends to a strongly connected modulo 3-orientation of G. And thus we will get a smaller counterexample. So the degree of each vertex of G must be odd and $|V(G)|$ can only be 4. By Lemma 2.4, G does not contain three parallel edges, and so each vertex v of G has exactly 3 neighbors. Thus G can only be isomorphic to the graph K_4^*, and then $G \in \mathcal{S}_3$ by Lemma 2.13, which is a contradiction. ■

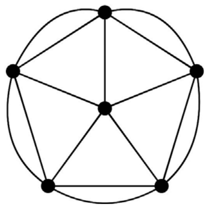

Fig. 2. The graph $2C_5 \cdot K_1$.

3. Proofs of the main results

Now we are ready to present the proofs of Theorems 1.7 and 1.10. In fact, we shall prove a stronger version of Theorem 1.10 with complete characterization of the bad attachments, stated as Theorem 3.3. In this section, *we always let G be a simple graph with* $\min\{\delta(G), \delta(G^c)\} \geq 4$, where G^c denotes the complement of G. For a vertex set $S \subset V(G)$, denote $\bar{S} = V(G) \setminus S$.

Lemma 3.1. *If G has an edge-cut of size at most* 3 *and* $|V(G)| \geq 26$, *then* $G^c \in \mathcal{S}_3$.

Proof. Let $E_G(S, \bar{S})$ be an edge-cut of size at most 3 in G. Since $\delta(G) \geq 4$, we have

$$|S|(|S|-1) \geq 2|E(G[S])| \geq 4|S| - e_G(S, \bar{S}) \geq 4|S| - 3,$$

which implies $|S| \geq 5$. Similarly, we have $|\bar{S}| \geq 5$ as well. Since $\frac{1}{2}|V(G)| \geq 13$, one of S and $\bar{S}$ has a size at least 13, say $|\bar{S}| \geq 13$.

In G^c, consider the subgraph $E_{G^c}(S, \bar{S})$. It is almost a complete bipartite graph with at most 3 edges deleted. Let $K_{s,t}$ be a maximal complete bipartite subgraph of $E_{G^c}(S, \bar{S})$ with $s = |S| \geq 5$. Then $t \geq |\bar{S}| - 3 \geq 10$. By Lemma 2.14, $K_{s,t} \in \mathcal{S}_3$. Let $S_1 = \{x \in \bar{S} | e_{G^c}(x, S) \leq 3\}$. Since $|S| \geq 5$ and $3 \geq e_G(\bar{S}, S) \geq e_G(S_1, S) \geq |S_1||S| - 3|S_1|$, we have $|S_1| \leq 1$. Note that $K_{s,t} (s \geq 5, t \geq 10)$ is 4-edge-connected, and $e_{G^c}(x, S) \geq 4$ for each $x \in \bar{S} \setminus S_1$. Thus $G^c - S_1 = G^c[\bar{S_1}]$ is 4-edge-connected. Since $\delta(G^c) \geq 4$, the only possible vertex in S_1 has at least 4 edges connecting $\bar{S_1}$. This implies that G^c is 4-edge-connected and $G^c = cl_3(K_{s,t})$. Thus $G^c \in \mathcal{S}_3$ by Lemma 2.6. ∎

Define

$$\mathcal{Y}_1 = \{Y \subseteq V(G) |\ \exists H \subseteq G \text{ with } H \in \mathcal{S}_3 \text{ and } G[Y] = cl_3(H) \text{ in } G\} \text{ and}$$

$$\mathcal{Y}_2 = \{Y \subseteq V(G) |\ \exists H \subseteq G^c \text{ with } H \in \mathcal{S}_3 \text{ and } G^c[Y] = cl_3(H) \text{ in } G^c\}.$$

$$\text{Choose } Y \in \mathcal{Y}_1 \cup \mathcal{Y}_2 \text{ with } |Y| \text{ maximized.} \tag{1}$$

Lemma 3.2. *If* $|V(G)| \geq 32$, *then* $|Y| \geq |V(G)| - 4$.

Proof. If $|V(G)| \geq 32$, then one of G, G^c has at least $\frac{1}{4}|V(G)|(|V(G)|-1) \geq 8(|V(G)|-1)$ edges. By Lemma 2.9, it contains a subgraph $H \in \mathcal{S}_3$ with $|V(H)| \geq 16$. Hence $|Y| \geq 16$ by (1). Without loss of generality, assume that $Y \in \mathcal{Y}_1$.

Suppose, to the contrary, that $|\bar{Y}| \geq 5$. Since $G[Y]$ is a 3-closure of an $\mathcal{S}_3$-graph in G, we have

$$e_G(Y, x) \leq 2 \text{ for each vertex } x \in \bar{Y}. \tag{2}$$

We first show the following statement:

$$\text{for any } Y_0 \in \mathcal{Y}_2, \text{ we have } \bar{Y} \not\subset Y_0. \tag{3}$$

If $\bar{Y} \subset Y_0$, then $\bar{Y_0} \subset Y$. For each $y \in \bar{Y_0}$, we have $e_{G^c}(y, Y_0) \leq 2$, and so $e_{G^c}(y, \bar{Y}) \leq 2$, which gives $e_G(y, \bar{Y}) \geq |\bar{Y}| - 2$. Hence, together with (2), we have

$$2|\bar{Y}| \geq e_G(Y, \bar{Y}) \geq (|\bar{Y}| - 2)|\bar{Y_0}|,$$

which implies that $|\bar{Y_0}| \leq 2|\bar{Y}|/(|\bar{Y}| - 2) < 4$ since $|\bar{Y}| \geq 5$ by the assumption. Hence $|Y_0| > |Y| + 1$, and it contradicts the maximality of $|Y|$ in (1). This proves (3).

Then we show the following statement:

$$|\bar{Y}| \geq 15. \tag{4}$$

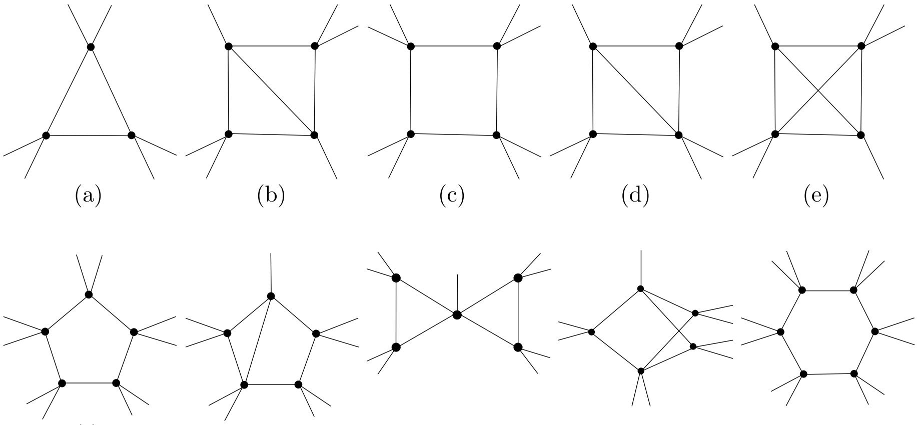

Fig. 3. Characterization of all bad attachments.

In fact, if $|\bar{Y}| < 15$, then $|Y| \geq 18$ as $|V(G)| \geq 32$. Let Z be a subset of $\bar{Y}$ with $|Z| = 4$. Denote $Y' = \{y \in Y | e_G(y, Z) = 0\}$. By (2), there are at most 8 vertices in Y that are adjacent to some vertices in Z. So $|Y'| \geq |Y| - 8 \geq 10$. This implies that $E_{G^c}(Y', Z)$ forms a complete bipartite graph $H_1 \cong K_{|Y'|,4} \in \mathcal{S}_3$ by Lemma 2.14.

Now in G^c, consider the 3-closure of H_1, namely $cl_3(H_1)$. We denote $Y_1 = V(cl_3(H_1))$ in G^c for convenience. By (2), for each vertex $x \in \bar{Y}$, we have $e_{G^c}(Y', x) = |Y'| - e_G(Y', x) \geq 10 - 2 > 3$, and so $x \in Y_1$ by definition. Thus $\bar{Y} \subset Y_1$. As $Y_1 = V(cl_3(H_1)) \in \mathcal{Y}_2$, it contradicts (3), and hence this proves (4).

Denote $X = \{y \in Y | e_G(y, \bar{Y}) \leq 1\}$. If $|X| \geq 5$, we let X_1 be a subset of X with $|X_1| = 5$. Let $Z_1 = \{z \in \bar{Y} | e_G(X_1, z) = 0\}$. Then in G there are at most 5 vertices in $\bar{Y}$ that are adjacent to some vertices in X_1. So $|Z_1| \geq |\bar{Y}| - 5 \geq 10$ by (4). Thus $E_{G^c}(X_1, Z_1)$ forms a complete bipartite graph $H_2 \cong K_{5,|Z_1|} \in \mathcal{S}_3$ by Lemma 2.14. Now consider the 3-closure of H_2 in G^c. Denote $Y_2 = V(cl_3(H_2))$. By (2), for each vertex $z \in \bar{Y}$, we have $e_{G^c}(X_1, z) = |X_1| - e_G(X_1, z) \geq 5 - 2 = 3$, and so $z \in Y_2$ by definition. This shows $\bar{Y} \subset Y_2$, a contradiction to (3). Thus we must have $|X| \leq 4$.

Since $|X| \leq 4$ and $|Y| \geq 16$, we let $y_1, y_2 \in Y \setminus X$ be two distinct vertices, that is, $e_G(y_i, \bar{Y}) \geq 2$ for each $i = 1, 2$. Denote by $u_i, v_i \in \bar{Y}$ the two distinct neighbors of y_i for each $i = 1, 2$. Let Z be a subset of $\bar{Y}$ with $|Z| = 4$ that contains $\{u_1, v_1\} \cup \{u_2, v_2\}$. Denote $Y' = \{y \in Y | e_G(y, Z) = 0\}$. Then $y_i \in Y \setminus Y'$ with $e_G(y_i, Z) \geq 2$ for $i = 1, 2$. By (2), we have

$$2|Z| \geq e_G(Y \setminus Y', Z) = \sum_{i=1}^{2} e_G(y_i, Z) + e_G((Y \setminus Y') \setminus \{y_1, y_2\}, Z) \geq 4 + (|Y \setminus Y'| - 2),$$

which implies that $|Y \setminus Y'| \leq 2|Z| - 2 = 6$, and so $|Y'| \geq |Y| - 6 \geq 10$.

Since $|Y'| \geq 10$, we have that $E_{G^c}(Y', Z)$ forms a complete bipartite graph $H_3 \cong K_{|Y'|,4} \in \mathcal{S}_3$ in G^c by Lemma 2.14. Consider the 3-closure of H_3 in G^c, and let $Y_3 = V(cl_3(H_3))$. By (2), for each vertex $x \in \bar{Y}$, we have $e_{G^c}(Y', x) = |Y'| - e_G(Y', x) \geq 10 - 2 > 3$, and hence $x \in Y_3$ by definition. Thus we have $\bar{Y} \subset Y_3$, which contradicts (3). This completes the proof of Lemma 3.2. ■

Proof of Theorem 1.7. By Lemma 3.1, we may assume that both G and G^c are 4-edge-connected. As in (1), we may, without loss of generality, assume that $Y \in \mathcal{Y}_1$. Thus $G[Y] = cl_3(H)$ for some subgraph $H \in \mathcal{S}_3$ in G. Then $G/G[Y]$ has at most 5 vertices by Lemma 3.2. Since $G/G[Y]$ is 4-edge-connected, we have $\phi(G/G[Y]) < 3$ by Lemma 2.16, and so $\phi(G) < 3$ by Lemma 2.6(i). This proves Theorem 1.7. ■

We shall prove the following theorem, which is stronger than Theorem 1.10. It provides a complete characterization of the bad attachments, and it also tells that the graph obtained by deleting bad attachment(s) is formed from the 3-closure of an $\mathcal{S}_3$-graph.

Theorem 3.3. *Let G be a simple graph with $|V(G)| \geq 77$. If $\min\{\delta(G), \delta(G^c)\} \geq 4$, then one of the following statements holds:*
(i) $G \in \mathcal{S}_3$ or $G^c \in \mathcal{S}_3$.
(ii) both G and G^c are formed from the 3-closure of an $\mathcal{S}_3$-subgraph by adding a bad attachment isomorphic to Fig. 3(c).
(iii) one of G and G^c is formed from the 3-closure of an $\mathcal{S}_3$-subgraph by adding a bad attachment isomorphic to Fig. 3(a); the other is formed from the 3-closure of an $\mathcal{S}_3$-subgraph by adding a bad attachment isomorphic to Fig. 3(a)–(j), or by adding two disjoint bad attachments isomorphic to Fig. 3(a).

Proof of Theorem 1.10 assuming Theorem 3.3. By Remark 2, we know that if G contains a bad attachment, then $G \notin \mathcal{S}_3$. Now it suffices to prove the "moreover part" of Theorem 1.10. Assume that both $G \notin \mathcal{S}_3$ and $G^c \notin \mathcal{S}_3$. Then both G

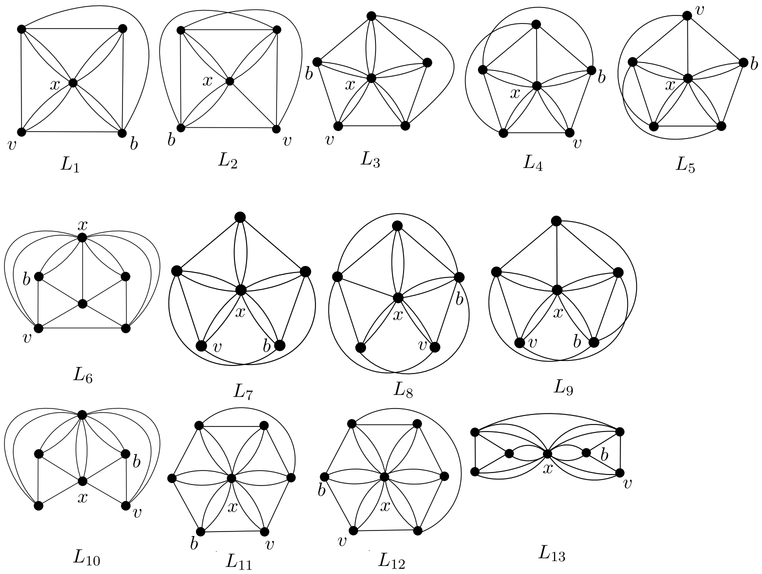

Fig. 4. A single edge added to each of the bad attachments.

and G^c are 4-edge-connected by Lemma 3.1. By Theorem 3.3, G is formed from the 3-closure of a subgraph $H \in \mathcal{S}_3$ by adding a bad attachment or two. By the description of the bad attachments in Fig. 3(a)–(j) in Theorem 3.3, $G/cl_3(H)$ is a 4-edge-connected graph on at most 5 vertices for Fig. 3(a)–(e), or $G/cl_3(H)$ is an Eulerian graph (i.e. every vertex has an even degree) for Fig. 3(f),(j) and for two disjoint bad attachments as Fig. 3(a), or $G/cl_3(H)$ is a 4-edge-connected graph with two odd vertices for Fig. 3(g),(h),(i). In each case, we have that $\phi(G/cl_3(H)) < 3$ by Lemma 2.16 or by constructing a strongly connected modulo 3-orientation. Thus $\phi(G) < 3$ by Lemma 2.6(i). The same proof works for G^c to show $\phi(G^c) < 3$. This finishes the proof of Theorem 1.10. ■

Before proving Theorem 3.3, we will show that some more graphs are in $\mathcal{S}_3$. Each of these graphs has only one more edge than the corresponding bad attachment, and any graph obtained from one of them by adding edges is in $\mathcal{S}_3$ by Observation 2.1.

Lemma 3.4. *Each of the graphs in Fig. 4 is in $\mathcal{S}_3$.*

Proof. For each $1 \le i \le 13$, let $G = L_i$ be a graph with $v, x, b \in V(G)$ as in Fig. 4. Then it is easy to check that $G_{[v,xb]}$ is 4-edge-connected and $G_{[v,xb]} = cl_3(x)$, and thus $G_{[v,xb]} \in \mathcal{S}_3$ by Lemma 2.6 (ii). It follows that $G \in \mathcal{S}_3$ from Lemma 2.11. ■

Proof of Theorem 3.3. By Lemma 3.1, we may assume that both G and G^c are 4-edge-connected. As in (1), we choose $Y \in \mathcal{Y}_1 \cup \mathcal{Y}_2$ with $|Y|$ maximized. Without loss of generality, assume $Y \in \mathcal{Y}_1$. Let $X = \{x \in Y | e_G(x, \bar{Y}) > 0\}$. Since Y is a 3-closure, for each vertex $x \in \bar{Y}$, $e_G(Y, x) \le 2$. Thus

$$|X| \le e_G(X, \bar{Y}) = e_G(Y, \bar{Y}) \le 2|\bar{Y}|. \tag{5}$$

Since $\delta(G) \ge 4$, we also have

$$4|\bar{Y}| - |\bar{Y}|(|\bar{Y}| - 1) \le e_G(Y, \bar{Y}) \le 2|\bar{Y}|, \tag{6}$$

which, together with Lemma 3.2, shows that $3 \le |\bar{Y}| \le 4$. We shall distinguish our discussion according to the value of $|\bar{Y}|$.

Case A. $|\bar{Y}| = 3$.

By (6), we have that $e_G(Y, \bar{Y}) = 6$ and $G[Y]$ forms a triangle. Thus this bad attachment of G is isomorphic to Fig. 3(a). It follows from (5) that $|X| \le 6$. Since $|V(G)| \ge 77$, we have $|Y \setminus X| \ge 68$.

In the complementary graph G^c, $E_{G^c}(Y \setminus X, \bar{Y})$ forms a complete bipartite graph $K_{3,|Y \setminus X|}$. Consider the subgraph $G^c[Y \setminus X]$ induced by $Y \setminus X$ in G^c. Let X_1 be the set of non-isolated vertices in $G^c[Y \setminus X]$. If $|X_1| \ge 14$, then $G^c[X_1 \cup \bar{Y}]$ forms a graph $H_1 \cong K^+_{3,|X_1|} \in \mathcal{S}_3$ by Lemma 2.15. Otherwise, we have $|X_1| \le 13$, which implies that there are at least 55 isolated vertices in $G^c[Y \setminus X]$. Since $\delta(G^c) \ge 4$ and $|\bar{Y}| = 3$, each isolated vertex in $G^c[Y \setminus X]$ is connected to X. Since $|X| \le 6$ and $|(Y \setminus X) \setminus X_1| \ge 55$, there exists a vertex $x_0 \in X$ such that $e_{G^c}(x_0, (Y \setminus X) \setminus X_1) \ge 10$ by Pigeon-Hole principle. Let

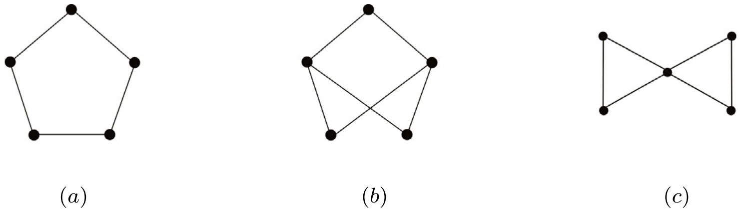

Fig. 5. $G^c[\bar{Z}]$ for $s = 5$.

$X_0 = \{y \in Y \setminus X | e_{G^c}(x_0, y) > 0\}$. Then $|X_0| \geq 10$ and $E_{G^c}(X_0, \bar{Y} \cup \{x_0\})$ forms a complete bipartite graph $H_2 = K_{4,|X_0|} \in \mathcal{S}_3$ by Lemma 2.14. Therefore, we can always find an $\mathcal{S}_3$-subgraph $H \in \{H_1, H_2\}$ in G^c that contains $\bar{Y}$. Now consider the 3-closure of H in G^c and let $Z = V(cl_3(H))$. Denote $s = |\bar{Z}|$. Since $E_{G^c}(Y \setminus X, \bar{Y})$ forms a complete bipartite graph $K_{3,|Y \setminus X|}$, we have $Y \setminus X \subset Z$, which is $\bar{Z} \subseteq X$. Then by (1),

$$3 = |\bar{Y}| \leq s \leq |X| \leq 6.$$

For each vertex $x \in \bar{Z}$, $e_{G^c}(Z, x) \leq 2$, and thus $e_{G^c}(Z, \bar{Z}) \leq 2s$. Since $\min\{\delta(G), \delta(G^c)\} \geq 4$ and $e_G(\bar{Z}, \bar{Y}) \leq e_G(Y, \bar{Y}) \leq 6$, we have $s(s-1) + e_{G^c}(Z, \bar{Z}) \geq 4s$ and $e_{G^c}(Z, \bar{Z}) \geq |\bar{Z}||\bar{Y}| - e_G(\bar{Z}, \bar{Y}) \geq 3s - 6$. In summary,

$$\max\{3s - 6, 5s - s^2\} \leq e_{G^c}(Z, \bar{Z}) \leq 2s. \tag{7}$$

Since $3 \leq s \leq 6$, we shall discuss the following cases, characterizing all the bad attachments in Theorem 3.3 (iii).

- $s = 3$.
  By (7), we have $e_{G^c}(Z, \bar{Z}) = 6$. Then the only possibility is that $\bar{Z}$ induces a bad attachment isomorphic to Fig. 3(a) in G^c.
- $s = 4$.
  Then $6 \leq e_{G^c}(Z, \bar{Z}) \leq 8$ by (7). If $e_{G^c}(Z, \bar{Z}) = 6$, then $\delta(G^c) \geq 4$ forces that the bad attachment induced by $\bar{Z}$ is isomorphic to Fig. 3(b) or (e).
  If $e_{G^c}(Z, \bar{Z}) = 7$, then $\delta(G^c) \geq 4$ forces that $G^c[\bar{Z}]$ has at least 5 edges. If $G^c[\bar{Z}] \cong K_4$, then $G^c/cl_3(H) \cong L_2 \in \mathcal{S}_3$ by Lemma 3.4, and so $G^c \in \mathcal{S}_3$ by Lemma 2.6(ii). Hence, Theorem 3.3(i) holds. Otherwise, $G^c[\bar{Z}]$ has exactly 5 edges, and the bad attachment induced by $\bar{Z}$ is isomorphic to Fig. 3(d).
  If $e_{G^c}(Z, \bar{Z}) = 8$, then $\delta(G^c) \geq 4$ implies that $G^c[\bar{Z}]$ contains a cycle C_4. If $G^c[\bar{Z}] \cong C_4$, then the bad attachment induced by $\bar{Z}$ is isomorphic to Fig. 3(c). Otherwise, $G^c[\bar{Z}]$ has at least 5 edges, and $G^c/cl_3(H)$ contains a subgraph $L_1 \in \mathcal{S}_3$ by Lemma 3.4. This shows that $G^c \in \mathcal{S}_3$ by Lemma 2.6(ii), and so Theorem 3.3(i) holds.
- $s = 5$.
  By (7), we have $9 \leq e_{G^c}(Z, \bar{Z}) \leq 10$, and $\delta(G^c) \geq 4$. This implies that $G^c[\bar{Z}]$ has minimum degree at least 2. So $G^c[\bar{Z}]$ contains one of the graphs C_5, $K_{2,3}$ and hourglass in Fig. 5 as a subgraph.
  If $e_{G^c}(Z, \bar{Z}) = 10$, then the bad attachment induced by $\bar{Z}$ is isomorphic to Fig. 3(f) when $G^c[\bar{Z}] \cong C_5$. Assume that $G^c[\bar{Z}]$ contains a cycle C_5 plus a chord. Then $G^c/cl_3(H)$ contains a subgraph $L_3 \in \mathcal{S}_3$ by Lemma 3.4. Therefore, $G^c \in \mathcal{S}_3$ by Lemma 2.6(ii), and so Theorem 3.3(i) holds. If $G^c[\bar{Z}]$ contains a subgraph isomorphic to (b) or (c) in Fig. 5, then $G^c/cl_3(H)$ has a subgraph isomorphic to L_7 or L_{10} in Fig. 4. Thus $G^c/cl_3(H) \in \mathcal{S}_3$ by Lemma 3.4, and so $G^c \in \mathcal{S}_3$ by Lemma 2.6(ii).
  If $e_{G^c}(Z, \bar{Z}) = 9$, then $\delta(G^c) \geq 4$ further forces that $G^c[\bar{Z}]$ contains a cycle C_5 plus a chord, or a subgraph isomorphic to (b) or (c) in Fig. 5. When $G^c[\bar{Z}]$ contains an additional edge, $G^c/cl_3(H)$ contains one of L_4, L_5, L_8, L_9 and L_6 in Fig. 4. All these graphs are in $\mathcal{S}_3$ by Lemma 3.4, and so $G^c \in \mathcal{S}_3$. Otherwise, the bad attachment induced by $\bar{Z}$ is isomorphic to Fig. 3(g), (h) or (i).
- $s = 6$.
  Then $e_{G^c}(Z, \bar{Z}) = 12$ by (7). Since $\delta(G^c) \geq 4$, $G^c[\bar{Z}]$ has minimum degree at least 2, and we deduce that $G^c[\bar{Z}]$ contains a C_6, two disjoint triangles, or a graph in Fig. 6(a)–(c).
  When $G^c[\bar{Z}]$ contains a graph in Fig. 6(a)–(c), $G^c/cl_3(H)$ contains a graph F in Fig. 6(i)-(iii). Since $F_{[v,xb]} = cl_3(x)$ and it is 4-edge-connected, we have $F_{[v,xb]} \in \mathcal{S}_3$ by Lemma 2.6. Then $F \in \mathcal{S}_3$ by Lemma 2.11, and so $G^c \in \mathcal{S}_3$ by Lemma 2.6. If $G^c[\bar{Z}]$ contains a cycle C_6 plus a chord, then $G^c/cl_3(H)$ contains L_{11} or $L_{12} \in \mathcal{S}_3$, and so $G^c \in \mathcal{S}_3$. If $G^c[\bar{Z}]$ contains two disjoint triangles plus an additional edge, then $G^c/cl_3(H)$ contains $L_{13} \in \mathcal{S}_3$. Thus $G^c \in \mathcal{S}_3$ and Theorem 3.3(i) holds. Otherwise, the bad attachment induced by $\bar{Z}$ is isomorphic to Fig. 3(j), or two disjoint bad attachments isomorphic to Fig. 3(a).

Case B. $|\bar{Y}| = 4$.

By (5), we have $|X| \leq 8$, and so $|Y \setminus X| = |Y| - |X| \geq 61 > 10$. Then in G^c, $E_{G^c}(Y \setminus X, \bar{Y})$ forms a complete bipartite graph $H \cong K_{4,|Y \setminus X|} \in \mathcal{S}_3$ by Lemma 2.14. Consider the 3-closure of H in G^c and let $Z = V(cl_3(H))$. Then $\bar{Y} \subset Z$ and $\bar{Z} \subseteq X$. For each vertex $x \in \bar{Z}$, we have $e_{G^c}(x, \bar{Y}) \leq e_{G^c}(x, Z) \leq 2$ by definition, and so

$$e_{G^c}(\bar{Z}, \bar{Y}) \leq 2|\bar{Z}|.$$

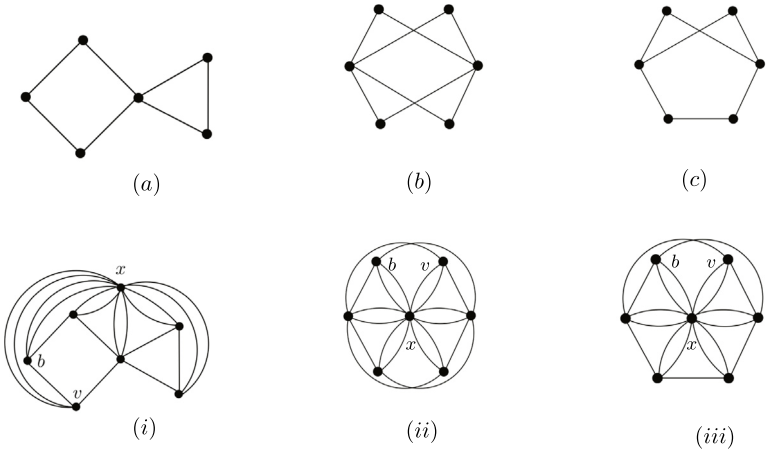

Fig. 6. The graphs for $s = 6$.

On the other hand, we have $e_G(\bar{Z}, \bar{Y}) \leq e_G(X, \bar{Y}) \leq 2|\bar{Y}| = 8$ by (5), and hence

$$e_{G^c}(\bar{Z}, \bar{Y}) = |\bar{Z}||\bar{Y}| - e_G(\bar{Z}, \bar{Y}) \geq 4|\bar{Z}| - 8.$$

Thus $4|\bar{Z}| - 8 \leq 2|Z|$, i.e., $|\bar{Z}| \leq 4$. By the maximality of Y in (1), we must have $|\bar{Z}| = 4$. Therefore, all the inequalities above are exactly equalities. Thus we have $e_G(Y, \bar{Y}) = e_G(\bar{Z}, \bar{Y}) = 8$ and $e_{G^c}(Z, \bar{Z}) = e_{G^c}(\bar{Y}, \bar{Z}) = 8$.

Now we will adapt the same argument as in the proof for $s = 4$ in Case A. Notice that $G[\bar{Y}]$ contains a cycle C_4 since $\delta(G) \geq 4$. If $G[\bar{Y}]$ has at least 5 edges, then $G/G[Y]$ contains a subgraph $L_1 \in \mathcal{S}_3$ by Lemma 3.4. This shows that $G \in \mathcal{S}_3$ by Lemma 2.6(ii), and so Theorem 3.3(i) holds. Otherwise, $G[\bar{Y}]$ is exactly a cycle C_4. Then in G the bad attachment induced by $\bar{Y}$ is isomorphic to Fig. 3(c). Analogously, either $G^c/cl_3(H) \in \mathcal{S}_3$ or the bad attachment of G^c induced by $\bar{Z}$ is isomorphic to Fig. 3(c). This completes the proof of Theorem 3.3. ■

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The authors are very grateful to the referee for many helpful comments and valuable suggestions, which led to an improvement of the presentation of this paper and corrections of several inaccuracies in the proof of Theorem 3.3. Jiaao Li is partially supported by National Natural Science Foundation of China (No. 11901318), Natural Science Foundation of Tianjin, China (No. 19JCQNJC14100) and the Fundamental Research Funds for the Central Universities, China, Nankai University (No. 63191425). Xueliang Li and Meiling Wang are partially supported by NSFC, China No. 11871034, 11531011 and NSFQH No. 2017-ZJ-790.

References

[1] L.A. Goddyn, M. Tarsi, C.-Q. Zhang, On (k, d)-colorings and fractional nowhere-zero flows, J. Graph Theory 28 (1998) 155–161.
[2] X. Hou, H.-J. Lai, P. Li, C.-Q. Zhang, Group connectivity of complementary graphs, J. Graph Theory 69 (2012) 464–470.
[3] F. Jaeger, Flows and generalized coloring theorems in graphs, J. Combin. Theory Ser. B 26 (1979) 205–216.
[4] F. Jaeger, in: L. Beineke, R. Wilson (Eds.), Nowhere-Zero Flow Problems, in: Selected Topics in Graph Theory, vol. 3, Academic Press, London/New York, 1988, pp. 91–95.
[5] J. Li, C. Thomassen, Y. Wu, C.-Q. Zhang, The flow index and strongly connected orientations, European J. Combin. 70 (2018) 164–177.
[6] L.M. Lovász, C. Thomassen, Y. Wu, C.-Q. Zhang, Nowhere-zero 3-flows and modulo k-orientations, J. Combin. Theory Ser. B 103 (2013) 587–598.
[7] W. Mader, A reduction method for edge-connectivity in graphs, Ann. Discrete Math. 3 (1978) 145–164.
[8] C.St.J.A. Nash-Williams, Decomposition of finite graphs into forest, J. Lond. Math. Soc. 39 (1964) 12.
[9] C. Thomassen, The weak 3-flow conjecture and the weak circular flow conjecture, J. Combin. Theory Ser. B 102 (2012) 521–529.
[10] X.J. Yao, X. Li, H.-J. Lai, Degree conditions for group connectivity, Discrete Math. 310 (2010) 1050–1058.
[11] D.H. Younger, Integer flows, J. Graph Theory 7 (1983) 349–357.
[12] C.-Q. Zhang, Integer Flows and Cycle Covers of Graphs, Marcel Dekker Inc., New York, ISBN: 0-8247-9790-6, 1997.