Discrete Mathematics 342 (2019) 1838–1848

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

List star edge coloring of k-degenerate graphs

Miaomiao Han [a], Jiaao Li [b], Rong Luo [c], Zhengke Miao [d,*]

[a] College of Mathematical Science, Tianjin Normal University, Tianjin 300387, People's Republic of China
[b] School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, People's Republic of China
[c] Department of Mathematics, West Virginia University , Morgantown, WV 26506, USA
[d] School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, Jiangsu, 221116, People's Republic of China

ARTICLE INFO

Article history:
Received 2 November 2017
Received in revised form 22 February 2019
Accepted 23 February 2019
Available online 18 March 2019

Keywords:
Star edge coloring
List star edge coloring
Tree
k-degenerate graph

ABSTRACT

A star edge coloring of a graph is a proper edge coloring such that every connected 2-colored subgraph is a path with at most 3 edges. Deng et al. and Bezegová et al. independently show that the star chromatic index of a tree with maximum degree Δ is at most $\lfloor \frac{3\Delta}{2} \rfloor$, which is tight. In this paper, we study the list star edge coloring of k-degenerate graphs. Let $ch'_{st}(G)$ be the list star chromatic index of G: the minimum s such that for every s-list assignment L for the edges, G has a star edge coloring from L. By introducing a stronger coloring, we show with a very concise proof that the upper bound on the star chromatic index of trees also holds for list star chromatic index of trees, i.e. $ch'_{st}(T) \leq \lfloor \frac{3\Delta}{2} \rfloor$ for any tree T with maximum degree Δ. And then by applying some orientation technique we present two upper bounds for list star chromatic index of k-degenerate graphs.

1. Introduction

Graphs in this paper are simple and finite. A star coloring of a graph is a proper vertex coloring such that the union of any two color classes induces a star forest. This notion was first introduced by Grünbaum [5] in 1973 and did not attract more attention until 2001 in the paper by Fertin, Raspaud and Reed [4]. Just like relation between concepts of traditional edge and vertex colorings, a star coloring of a line graph is a star edge coloring of the original graph.

A *star edge coloring* of a graph G is a proper edge coloring such that every connected bicolored subgraph is a path of length at most 3 (the length of a path is the number of edges). The notion of the star edge coloring is intermediate between acyclic edge coloring, when every bicolored subgraph is acyclic, and strong edge coloring when every bicolored connected subgraph has at most two edges.

The star chromatic index of G, denoted by $\chi'_{st}(G)$, is the smallest integer k such that G is star k-edge-colorable. Liu and Deng [11] showed that $\chi'_{st}(G) \leq \lceil 16(\Delta - 1)^{\frac{3}{2}} \rceil$ when $\Delta \geq 7$. Dvořák, Mohar, and Šámal [3] presented a near-linear upper bound for $\chi'_{st}(G)$.

Theorem 1.1 ([3]). *For any graph G with maximum degree Δ, $\chi'_{st}(G) \leq \Delta \cdot 2^{O(1)\sqrt{\log \Delta}}$.*

Bezegová et al. [1] and Deng et al. [2] independently proved the following bound for trees.

* Corresponding author.
E-mail addresses: mmhan2018@hotmail.com (M. Han), lijiaao@nankai.edu.cn (J. Li), rluo@math.wvu.edu (R. Luo), zkmiao@jsnu.edu.cn (Z. Miao).

https://doi.org/10.1016/j.disc.2019.02.018

Theorem 1.2 (*[1,2]*). *Let T be a tree with maximum degree Δ. Then*

$$\chi'_{st}(T) \le \left\lfloor \frac{3\Delta}{2} \right\rfloor,$$

and the bound is tight.

It seems very difficult to determine the star chromatic index of graphs even for complete graphs and subcubic graphs. Lei, Shi, and Song [9] showed that it is NP-complete to determine whether a subcubic multigraph is star 3-edge-colorable. Dvořák, Mohar, and Šámal [3] presented the following upper and lower bounds for complete graphs:

$$2n(1+o(1)) \le \chi'_{st}(K_n) \le n\frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log n}}}{(\log n)^{\frac{1}{4}}}.$$

Dvořák, Mohar, and Šámal [3] also studied star edge coloring of subcubic graphs and proved the following.

Theorem 1.3 (*[3]*). *If G is a subcubic graph, then $\chi'_{st}(G) \le 7$.*

They made the following conjecture.

Conjecture 1.4 (*[3]*). *If G is a subcubic graph, then $\chi'_{st}(G) \le 6$.*

A natural generalization of star edge coloring is the list star edge coloring and it was pointed out in [3]: It would be interesting to understand the list version of star edge coloring.

For a given list assignment L which assigns to each edge e a finite set $L(e)$, a graph is said to be L-star-edge-colorable if G has a star edge coloring c such that $c(e) \in L(e)$ for each edge e. L is called an edge k-list if each $L(e)$ is a set of size at least k. A graph G is star k-edge-choosable if for any edge k-list L there is a star edge coloring c such that $c(e) \in L(e)$ for every edge e. The list star chromatic index of a graph G, denoted by $ch'_{st}(G)$, is the minimum k such that G is star k-edge-choosable.

Dvořák, Mohar, and Šámal proposed the following problem in [3] for the list star edge coloring.

Problem 1.5 (*[3]*). Is it true that $ch'_{st}(G) \le 7$ for every subcubic graph G? (Perhaps even ≤ 6).

Problem 1.6 (*[3]*). Is it true that $ch'_{st}(G) = \chi'_{st}(G)$ for every graph G?

In an attempt to solve Problem 1.5, Kerdjoudj et al. [6] proved the following results on list version for subcubic graphs with certain maximum average degree conditions. The *maximum average degree* of a graph G is defined by $mad(G) = max\{\frac{2|E(H)|}{|V(H)|} : H \subseteq G\}$.

Theorem 1.7 (*[6]*). *Let G be a subcubic graph. Then each of the following holds.*
(i) $ch'_{st}(G) \le 8$.
(ii) If $mad(G) < \frac{7}{3}$, then $ch'_{st}(G) \le 5$.
(iii) If $mad(G) < \frac{5}{2}$, then $ch'_{st}(G) \le 6$.

In this paper, we attempt to study the list star edge coloring of general graphs and present a couple of upper bounds on the list star chromatic index in terms of degeneracy. After this paper was submitted, there are quite a few papers on this topic are published (c.f.[7,8,10,13]). Also, Problem 1.5 was solved by Lužar, Mockovčiaková and R. Soták [12].

By introducing the notion of a slightly stronger edge coloring (than star edge coloring) we first give a concise proof for the list star chromatic index of trees, and thus extend Theorem 1.2 to the list star chromatic index. Then by modifying the ideas of the proof for trees and introducing some orientation technique, we present some upper bounds on list star chromatic index of k-degenerate graphs for general $k \ge 2$ (Theorems 1.8 and 1.9). Besides the orientation technique, our main coloring strategy is to find a partition of each $E(v)$ into two parts such that the colors used by the edges in one part can be repeated by some edges with distance two from them. This will help reduce the number of forbidden colors. We believe that our method will be useful in the future study of star edge coloring.

Theorem 1.8. *For every tree T with maximum degree Δ,*

$$ch'_{st}(T) \le \left\lfloor \frac{3\Delta}{2} \right\rfloor,$$

and this bound is tight.

Theorem 1.9. *Let $k \ge 2$ be an integer. For every k-degenerate graph G with maximum degree Δ, we have the following two upper bounds:*

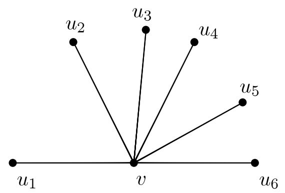

Fig. 1. An example on definition of distance.

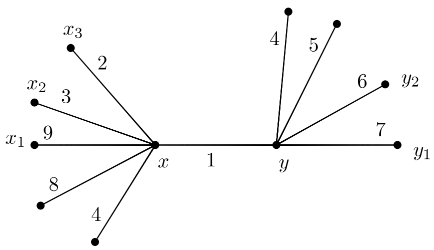

Fig. 2. A $\frac{1}{2}$-strong 9 edge coloring: $c(x_1x), c(x_2x), c(x_3x) \notin c(y)$ and $c(y_1y), c(y_2y) \notin c(x)$.

(a) $ch'_{st}(G) \leq \frac{5k-1}{2}\Delta - \frac{k(k+3)}{2}$. *The bound is tight for* C_5 *as* $ch'_{st}(C_5) = 4$.
(b) $ch'_{st}(G) \leq 2k\Delta + k^2 - 4k + 2$.

Remark 1. By comparing those two bounds, it is easy to see that (a) is better than (b) if and only if $k \leq \frac{\Delta}{3}$.

Remark 2. Theorem 1.8 implies that if $\chi'_{st}(T) = \lfloor \frac{3\Delta}{2} \rfloor$, then $\chi'_{st}(T) = ch'_{st}(T)$. In particular, it is proved in [1] and [2] that if T is a tree which has a Δ-vertex whose neighbors are all Δ-vertices, then $\chi'_{st}(T) = \lfloor \frac{3\Delta}{2} \rfloor$ and thus $\chi'_{st}(T) = ch'_{st}(T) = \lfloor \frac{3\Delta}{2} \rfloor$ by Theorem 1.8. This responds to Problem 1.6 for some trees.

Before proceeding we need to introduce some notation. A graph is *k-degenerate* if every subgraph has minimum degree at most k. A connected graph is 1-degenerate if and only if it is a tree. We use $V(G)$ and $E(G)$ to denote the vertex set and edge set in a graph G, respectively. For each integer $k \geq 1$, let $[k] = \{1, 2 \cdots, k\}$ and denote $V_k(G)$, $V_{\leq k}(G)$, $V_{\geq k+1}(G)$ the set of vertices with degree k, at most k, at least $k+1$ in G respectively. Denote $E_G(u)$ the set of edges incident with the vertex u in G and $d_G(u) = |E_G(u)|$ the degree of u in G. Let D be an orientation of G. For each vertex $v \in V(G)$, denote $E_D^+(v)$ ($E_D^-(v)$, respectively) to be the edges oriented out from (into, respectively) the vertex v, and let $d^-(v) = |E_D^-(v)|$ and $d^+(v) = |E_D^+(v)|$.

2. Star edge coloring and $\frac{1}{2}$-strong edge coloring

We first apply our coloring strategy on trees, and then we generalize it to arbitrary graphs.

2.1. List star edge coloring and list $\frac{1}{2}$-strong edge coloring on trees

In this subsection we will prove Theorem 1.8.

Let G be a planar graph embedded on the plane. For each pair of adjacent edges $u_1v, u_2v \in E(v)$, define the distance from u_1v to u_2v at v to be

$$\rho_v(u_1v, u_2v) = 1 + |\{uv \in E(v) : u_1v, uv, u_2v \text{ are located in the clockwise order}\}|.$$

It is obvious that $\rho_v(u_1v, u_2v) + \rho_v(u_2v, u_1v) = d_G(v)$.

Example: In Fig. 1, $\rho_v(u_1v, u_2v) = 1$, $\rho_v(u_2v, u_1v) = 5$, $\rho_v(u_3v, u_5v) = 2$, $\rho_v(u_6v, u_3v) = 3$.

For an edge coloring c and each vertex x, denote $c(x) = \{c(xu) : xu \in E(G)\}$.

Definition 2.1. Let G be a plane graph and $0 \leq r \leq 1$ be a rational number. An *r-strong edge coloring* of G is an edge coloring $c : E(G) \mapsto [k]$ such that

(i) $c(e_1) \neq c(e_2)$ for any two adjacent edges e_1, e_2;
(ii) for any edge $xy \in E(G)$, if $\rho_x(vx, yx) \leq rd_G(x)$, then $c(vx) \notin c(y)$; if $\rho_y(uy, xy) \leq rd_G(y)$, then $c(uy) \notin c(x)$.

A 0-strong edge coloring is a proper edge coloring, and a 1-strong edge coloring is a strong edge coloring. In this paper, we focus on $\frac{1}{2}$-strong edge coloring of graphs. We first show that a $\frac{1}{2}$-strong edge coloring is always a star edge coloring and then show that every tree T with maximum degree Δ has a list $\frac{1}{2}$-strong edge coloring as long as $|L(e)| \ge \lfloor\frac{3\Delta}{2}\rfloor$ for each edge e. An example of 1/2-strong edge coloring can be seen in Fig. 2.

Lemma 2.2. *Let G be a plane graph and c be a proper edge coloring of G. If c is a $\frac{1}{2}$-strong edge coloring, then c is a star edge coloring of G.*

Proof. Suppose to the contrary that c is not a star edge coloring. Let $P = xyzuv$ be a bicolored path (or cycle) where $c(xy) = c(zu)$ and $c(yz) = c(uv)$. By the definition of $\frac{1}{2}$-strong edge coloring, we have $c(tz) \ne c(xy)$ for any $tz \in E(z)$ with $\rho_z(tz, yz) \le \frac{1}{2}d_G(z)$. Thus $\rho_z(uz, yz) \ge \lfloor\frac{d_G(z)}{2}\rfloor + 1$ since $c(uz) = c(xy) \in c(y)$. For the same reason, we have $\rho_z(yz, uz) \ge \lfloor\frac{d_G(z)}{2}\rfloor + 1$. This implies

$$d_G(z) = \rho_z(yz, uz) + \rho_z(uz, yz) \ge \left\lfloor\frac{d_G(z)}{2}\right\rfloor + 1 + \left\lfloor\frac{d_G(z)}{2}\right\rfloor + 1 \ge d_G(z) + 1,$$

a contradiction. ■

Now we are ready to prove our result on trees (Theorem 1.8). By Lemma 2.2, Theorem 1.8 follows directly from the theorem below.

Theorem 2.3. *Let T be a tree with maximum degree Δ embedded on the plane and L be a $\lfloor\frac{3\Delta}{2}\rfloor$-list assignment. Then there exists a $\frac{1}{2}$-strong edge coloring c such that $c(e) \in L(e)$ for every $e \in E(G)$.*

Proof. We prove the theorem by induction on $|V(T)|$. The theorem is obvious if $|V(T)| = 2$. We assume $|V(T)| \ge 3$. Let x be a vertex in T such that x is adjacent to at least $d_T(x) - 1$ leaves. Denote $t = d_T(x) - 1$ and let $x_1x, \dots, x_tx, yx$ be the edges in $E_T(x)$ in counterclockwise order where $x_1, x_2, \dots, x_t$ are leaves. Let $T' = T - \{x_1, \dots, x_t\}$. By induction hypothesis, T' has a $\frac{1}{2}$-strong edge coloring c' such that $c'(e) \in L(e)$ for every $e \in E(T')$. We shall extend c' to be a $\frac{1}{2}$-strong edge coloring c of T.

Denote $s = \lfloor\frac{d_T(x)}{2}\rfloor$. For every $1 \le i \le s$, we have

$$|L(x_ix) \setminus c'(y)| \ge \left\lfloor\frac{3\Delta}{2}\right\rfloor - \Delta \ge s.$$

Thus we can first color the edges $x_1x, \dots, x_sx$ properly by coloring each x_ix with a color from $L(x_ix) \setminus c'(y)$ for every $1 \le i \le s$.

Denote $l = \lfloor\frac{d_T(y)}{2}\rfloor$. Let $y_1, \dots, y_l$ be all the neighbors of y with $\rho_y(y_jy, xy) \le l$ ($j \in [l]$) and denote $L_0 = \{c(x_ix) : i \in [s]\} \cup \{c(y_jy) : j \in [l]\} \cup \{c(xy)\}$. By the definition of $\frac{1}{2}$-strong edge coloring, L_0 is the set of all forbidden colors for xx_j for each $s + 1 \le j \le t$.

Then for each $s + 1 \le j \le t$, we have

$$|L(xx_j) \setminus L_0| \ge \left\lfloor\frac{3\Delta}{2}\right\rfloor - \left\lfloor\frac{\Delta}{2}\right\rfloor - 1 - \left\lfloor\frac{\Delta}{2}\right\rfloor = \Delta - 1 - \left\lfloor\frac{\Delta}{2}\right\rfloor \ge t - s.$$

Thus we can color the edges $x_{s+1}x, x_{s+2}x, \dots, x_tx$ properly by coloring x_jx with a color from $L(xx_j) \setminus L_0$ for each $s + 1 \le j \le t$.

Finally, we show this coloring is a $\frac{1}{2}$-strong edge coloring of T. It suffices to verify the edge xy satisfying condition (ii) of Definition 2.1. Let $v \in \{x_1, \dots, x_s\}$ and $u \in \{y_1, \dots, y_l\}$. If $\rho_x(vx, xy) \le \lfloor\frac{1}{2}d_T(x)\rfloor = s$, we have $c(vx) \notin c(y)$; and if $\rho_y(uy, xy) \le \lfloor\frac{1}{2}d_T(y)\rfloor = l$, we have $c(uy) \notin c(x)$. Therefore, the resulting coloring c is a $\frac{1}{2}$-strong edge coloring of T. The proof is completed. ■

2.2. A generalization of $\frac{1}{2}$-strong edge coloring

Note that in the definition of $\frac{1}{2}$-strong edge coloring of a plane graph G, we only use the clockwise order of $E(v)$ for each vertex v, but not any other planarity structures. So the idea of $\frac{1}{2}$-strong edge coloring can be generalized to arbitrary graphs as long as we have a cyclic ordering of edges in $E(v)$ for each vertex v.

Definition 2.4. Let G be a graph and let $\sigma(v)$ be a cyclic ordering of the edges in $E(v)$ for each vertex v. σ is called a local ordering of $E(G)$. The distance from edge uv to wv at v with respect to σ, denoted by $\rho_{\sigma,v}(uv, wv)$, is their distance in $\sigma(v)$.

One may consider $\sigma(v)$ as a directed cycle with vertex set $E(v)$ and the distance from uv to wv is the length of the directed path from uv to wv in the directed cycle. Thus $\rho_{\sigma,v}(uv, wv) + \rho_{\sigma,v}(wv, uv) = d(v)$. Denote

$$F_{\sigma,v}(uv) = \{wv \in E(v) : \rho_{\sigma,v}(uv, wv) \leq \left\lfloor \frac{d(v)}{2} \right\rfloor\}.$$

Let G be a graph and σ be a local ordering of $E(G)$. A proper edge coloring c is a $\frac{1}{2}$-*strong edge coloring* with respect to σ provided that for each edge $uv \in E(G)$, $c(uv) \notin c(w)$ if $wv \in F_{\sigma,v}(uv)$ (or equivalently $\rho_{\sigma,v}(uv, wv) \leq \lfloor \frac{d(v)}{2} \rfloor$).

For convenience, the local ordering σ will be mentioned explicitly only when needed. If σ is understood from the context, we simply use $\rho_v(uv, wv)$ and $F_v(uv)$ to denote $\rho_{\sigma,v}(uv, wv)$ and $F_{\sigma,v}(uv)$, respectively. Note $|F_v(uv)| = \lfloor \frac{d(v)}{2} \rfloor$.

Similar to Lemma 2.2, a $\frac{1}{2}$-strong edge coloring c of G with respect to σ is a star edge coloring.

Lemma 2.5. *Let G be a graph. For any local ordering, every $\frac{1}{2}$-strong edge coloring of G is a star edge coloring.*

Proof. Suppose to the contrary that $P = xyzuv$ is a bicolored path (or cycle) of length four in a $\frac{1}{2}$-strong edge coloring c of G. Since $c(yz) \in c(u)$, we have $\rho_z(yz, uz) > \lfloor \frac{d(z)}{2} \rfloor$. Since $c(zu) \in c(y)$, $\rho_z(uz, yz) > \lfloor \frac{d(z)}{2} \rfloor$. Thus $\rho_z(uz, yz) + \rho_z(yz, uz) \geq 2(\lfloor \frac{d(z)}{2} \rfloor + 1) > d(z)$, a contradiction to the fact that $\rho_z(uz, yz) + \rho_z(yz, uz) = d_G(z)$. ■

We show a general upper bound on the list $\frac{1}{2}$-strong edge coloring of graphs, which will be needed in the proof of Theorem 1.9-(b) when $k \in \{\Delta, \Delta - 1, \Delta - 2\}$.

For two positive integers Δ and k, denote

$$\ell = \begin{cases} \frac{3}{4}\Delta^2 + (k-1)\Delta, & \text{if } k \leq \lfloor \frac{\Delta}{2} \rfloor \text{ and } \Delta \text{ is even;} \\ \frac{3}{4}\Delta^2 + \frac{2k-3}{2}\Delta + \frac{3}{4}, & \text{if } k \leq \lfloor \frac{\Delta}{2} \rfloor \text{ and } \Delta \text{ is odd;} \\ \Delta^2 + \frac{k-4}{2}\Delta + 2k - 1, & \text{if } k \geq \lfloor \frac{\Delta}{2} \rfloor + 1 \text{ and } \Delta \text{ is even;} \\ \Delta^2 + \frac{k-5}{2}\Delta + \frac{3k+3}{2}, & \text{if } k \geq \lfloor \frac{\Delta}{2} \rfloor + 1 \text{ and } \Delta \text{ is odd.} \end{cases}$$

Theorem 2.6. *Let G be a k-degenerate graph with maximum degree $\Delta \geq 3$. Then, for any local ordering and for any ℓ-list assignment L, there exists a $\frac{1}{2}$-strong edge coloring c such that $c(e) \in L(e)$ for every $e \in E(G)$.*

Proof. Let σ be a local ordering of $E(G)$. Let G be a counterexample with $|E(G - V_1)|$ minimized. By Theorem 1.2, G is not a tree and $G - V_1$ is connected. Let v be a vertex such that $d_{G-V_1}(v)$ is the minimum in $G - V_1$. Denote $E_G(v) = \{x_1v, \ldots, x_tv, y_1v, \ldots, y_sv\}$, where $d_G(x_i) \geq 2$ and $d_G(y_j) = 1$ for each $1 \leq i \leq t$ and each $1 \leq j \leq s$. Construct a new graph G' from $G - v$ by adding new degree one vertex x_i' connecting x_i for each $1 \leq i \leq t$ where the edge $x_i'x_i$ plays the same role as vx_i in the ordering $\sigma(x_i)$. Since v is adjacent to at least one vertex of degree large than one in G, we have $|E(G' - V_1(G'))| < |E(G - V_1)|$. By the minimality of G, there exists a $\frac{1}{2}$-strong edge coloring c' such that $c'(e) \in L(e)$ for every $e \in E(G')$. Uncolor the edges $x_i'x_i$'s and we still use c' to denote the new coloring. Then the coloring c' restricted to $G - v$ is a partial $\frac{1}{2}$-strong edge coloring of G, and we shall extend c' to a $\frac{1}{2}$-strong edge coloring c of G by coloring the edges in $E(v)$ appropriately.

We color the edges x_iv in $\{x_1v, x_2v, \ldots, x_tv\}$ with $|F_v(x_iv) \cap \{x_1v, x_2v, \ldots, x_tv\}| = \lfloor \frac{\Delta}{2} \rfloor$ first, and then color the remaining edges in $\{x_1v, x_2v, \ldots, x_tv\}$, and finally we color the edges $y_1v, \ldots, y_sv$.

In the following, we estimate the maximum number of forbidden colors in order to color the edges in $E(v)$. Let $uv \in E(v)$ where $u \in \{x_1, \ldots, x_t\}$. Suppose we pick a color α to color uv.

We first consider the forbidden colors on u's side. By the definition of $\frac{1}{2}$-strong edge coloring, we have

(i) for each edge $uw \in F_u(vu)$, $\alpha \notin c'(w)$. Since $|c'(w)| = d_G(w) \leq \Delta$ and there are $|F_u(vu)|$ such edges, the total number of forbidden colors from those edges is at most $|F_u(vu)|\Delta = \lfloor \frac{d(u)}{2} \rfloor \Delta$;

(ii) for each edge $zu \notin F_u(vu)$ and for any $z'z \in E(G)$ with $uz \in F_z(z'z)$, $c'(z'z)$ does not appear in $c'(u)$. Since $c'(z'z) \notin c'(u)$, we have $\alpha \neq c'(z'z)$ and thus including $c'(zu)$, there are at most $\lfloor \frac{\Delta}{2} \rfloor + 1$ forbidden colors in $c'(z)$. Since uv is not colored yet, there are $(d(u) - 1 - \lfloor \frac{d(u)}{2} \rfloor)$ such edges zu. Therefore the total number of forbidden colors from those edges is at most $(d(u) - 1 - \lfloor \frac{d(u)}{2} \rfloor)(\lfloor \frac{\Delta}{2} \rfloor + 1)$.

So the number of forbidden colors on u's side is at most

$$\left\lfloor \frac{d(u)}{2} \right\rfloor \Delta + (d(u) - 1 - \left\lfloor \frac{d(u)}{2} \right\rfloor)(\left\lfloor \frac{\Delta}{2} \right\rfloor + 1) \leq \left\lfloor \frac{\Delta}{2} \right\rfloor \Delta + (\Delta - 1 - \left\lfloor \frac{\Delta}{2} \right\rfloor)(\left\lfloor \frac{\Delta}{2} \right\rfloor + 1). \tag{1}$$

Now we consider the forbidden colors on v's side. Note that $y_1v, \ldots, y_sv$ are not colored yet. It is clear that the number of forbidden colors on v's side is at most

$$(t-1)\Delta \leq (k-1)\Delta. \tag{2}$$

However we can have better estimation when $t \geq \lfloor \frac{\Delta}{2} \rfloor + 1$.

Denote $A = F_v(uv) \cap \{x_1v, x_2v, \dots, x_tv\}$ and $a = |A|$. Let h be the number of colored edges in $F_v(uv)$, and let $u'v$ be the colored edge in A with $\rho_v(uv, u'v)$ maximized.

Similar to (i) and (ii) we have the following:

(iii) For each edge $wv \in A$, $\alpha \notin c'(w)$ and thus there are $d_G(w) \le \Delta$ or $d_G(w) - 1 \le \Delta - 1$(depending on whether wv is already colored or not) forbidden colors at w.

(iv) For each edge $wv \in \{x_1v, x_2v, \dots, x_tv\} - F_v(uv)$, similar to (ii) there are at most $\lfloor \frac{\Delta}{2} \rfloor + 1$ forbidden colors. Note there are at most $(t - 1 - a)$ such edges.

If $a = |A| \le \lfloor \frac{\Delta}{2} \rfloor - 1$, by (iii) and (iv) the total number of forbidden colors caused by v's side is at most

$$\begin{aligned}
& a\Delta + (t-1-a)\left(\left\lfloor \frac{\Delta}{2} \right\rfloor + 1\right) \\
\le\ & \left(\left\lfloor \frac{\Delta}{2} \right\rfloor - 1\right)\Delta + \left(t - \left\lfloor \frac{\Delta}{2} \right\rfloor\right)\left(\left\lfloor \frac{\Delta}{2} \right\rfloor + 1\right) \\
\le\ & \left\lfloor \frac{\Delta}{2} \right\rfloor\left(\Delta - \left\lfloor \frac{\Delta}{2} \right\rfloor - 1\right) - \Delta + k\left(\left\lfloor \frac{\Delta}{2} \right\rfloor + 1\right) \\
\le\ & \left\lfloor \frac{\Delta}{2} \right\rfloor\left(\Delta - \left\lfloor \frac{\Delta}{2} \right\rfloor - 2\right) + k\left(\left\lfloor \frac{\Delta}{2} \right\rfloor + 2\right) - \Delta - 1 \qquad (\text{by } \lfloor \tfrac{\Delta}{2} \rfloor + 1 \le k) \\
\le\ & \left\lfloor \frac{\Delta}{2} \right\rfloor\left(\Delta - \left\lfloor \frac{\Delta}{2} \right\rfloor - 4\right) + k\left(\left\lfloor \frac{\Delta}{2} \right\rfloor + 2\right) - 1 \qquad (\text{by } 2\lfloor \tfrac{\Delta}{2} \rfloor \le \Delta).
\end{aligned}$$

Now assume $|A| = \lfloor \frac{\Delta}{2} \rfloor$. Then $F_v(uv) \subseteq \{x_1v, x_2v, \dots, x_tv\}$ and $\lfloor \frac{d_G(v)}{2} \rfloor = \lfloor \frac{\Delta}{2} \rfloor$. Since $u'v$ is already colored, by the coloring algorithm, $|F_v(u'v) \cap \{x_1v, x_2v, \dots, x_tv\}| = |A| = \lfloor \frac{\Delta}{2} \rfloor$. Thus $F_v(u'v) \subseteq \{x_1v, x_2v, \dots, x_tv\}$ and $|F_v(u'v)| = \lfloor \frac{\Delta}{2} \rfloor$. Note that $h \le \rho_v(uv, u'v)$. Since the colored edges in $F_v(uv)$ do not belong to $F_v(u'v)$ if $h \ne 0$, we have $h + \lfloor \frac{\Delta}{2} \rfloor \le \rho_v(uv, u'v) + |F_v(u'v)| \le t$. Thus

$$h \le t - \left\lfloor \frac{\Delta}{2} \right\rfloor. \tag{3}$$

By (iii) and (iv), the total number of forbidden colors on v's side is at most

$$\begin{aligned}
& h\Delta + \left(\left\lfloor \frac{\Delta}{2} \right\rfloor - h\right)(\Delta - 1) + \left(t - 1 - \left\lfloor \frac{\Delta}{2} \right\rfloor\right)\left(\left\lfloor \frac{\Delta}{2} \right\rfloor + 1\right) \\
=\ & \left\lfloor \frac{\Delta}{2} \right\rfloor(\Delta - 1) + h + \left(t - 1 - \left\lfloor \frac{\Delta}{2} \right\rfloor\right)\left(\left\lfloor \frac{\Delta}{2} \right\rfloor + 1\right) \\
\le\ & \left\lfloor \frac{\Delta}{2} \right\rfloor(\Delta - 1) + t - \left\lfloor \frac{\Delta}{2} \right\rfloor + \left(t - 1 - \left\lfloor \frac{\Delta}{2} \right\rfloor\right)\left(\left\lfloor \frac{\Delta}{2} \right\rfloor + 1\right) \quad (\text{by Inequality (3)}) \\
=\ & \left\lfloor \frac{\Delta}{2} \right\rfloor(\Delta - 2) + t\left(\left\lfloor \frac{\Delta}{2} \right\rfloor + 2\right) - \left(1 + \left\lfloor \frac{\Delta}{2} \right\rfloor\right)^2 \\
\le\ & \left\lfloor \frac{\Delta}{2} \right\rfloor(\Delta - 2) + k\left(\left\lfloor \frac{\Delta}{2} \right\rfloor + 2\right) - \left(1 + \left\lfloor \frac{\Delta}{2} \right\rfloor\right)^2 \qquad (\text{by } t \le k) \\
=\ & \left\lfloor \frac{\Delta}{2} \right\rfloor\left(\Delta - \left\lfloor \frac{\Delta}{2} \right\rfloor - 4\right) + k\left(\left\lfloor \frac{\Delta}{2} \right\rfloor + 2\right) - 1.
\end{aligned}$$

Therefore, if $k \ge \lfloor \frac{\Delta}{2} \rfloor + 1$, then by Inequality (1) the total number of forbidden colors for uv is at most

$$\left\lfloor \frac{\Delta}{2} \right\rfloor \Delta + \left(\Delta - 1 - \left\lfloor \frac{\Delta}{2} \right\rfloor\right)\left(\left\lfloor \frac{\Delta}{2} \right\rfloor + 1\right) + \left\lfloor \frac{\Delta}{2} \right\rfloor\left(\Delta - \left\lfloor \frac{\Delta}{2} \right\rfloor - 4\right) + k\left(\left\lfloor \frac{\Delta}{2} \right\rfloor + 2\right) - 1 \le \ell - 1.$$

If $k \le \lfloor \frac{\Delta}{2} \rfloor$, then by Inequalities (1) and (2) the total number of forbidden colors for uv is at most

$$\left\lfloor \frac{\Delta}{2} \right\rfloor \Delta + \left(\Delta - 1 - \left\lfloor \frac{\Delta}{2} \right\rfloor\right)\left(\left\lfloor \frac{\Delta}{2} \right\rfloor + 1\right) + (k-1)\Delta \le \ell - 1.$$

Finally, when we color y_jv ($j \in [s]$), the total number of forbidden colors is at most

$$\left\lfloor \frac{\Delta}{2} \right\rfloor \Delta + \left(\Delta - 1 - \left\lfloor \frac{\Delta}{2} \right\rfloor\right)\left(\left\lfloor \frac{\Delta}{2} \right\rfloor + 1\right) \le \ell - 1.$$

Therefore, we can complete the coloring process to obtain a $\frac{1}{2}$-strong edge coloring c of G, a contradiction. This completes the proof of the theorem. ■

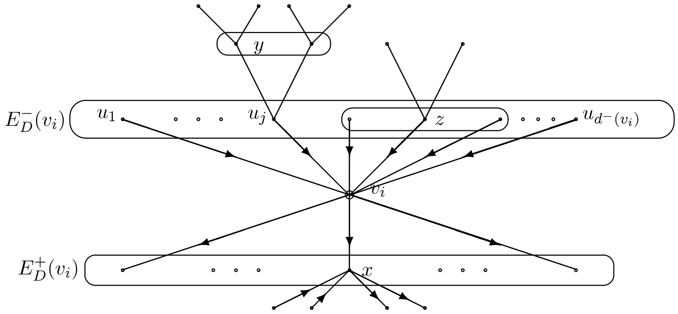

Fig. 3. Local structure of $E(v_i)$.

Note that Theorem 2.6 also provides a general upper bound $\frac{3}{2}\Delta^2 - 1$ (and $\frac{3}{2}\Delta^2 - \Delta + \frac{3}{2}$ when Δ is odd) for $\frac{1}{2}$-strong edge coloring of graphs with maximum degree Δ.

3. List star edge coloring of k-degenerate graphs—two more upper bounds

In this section, we modify the idea of the proof of trees by introducing a special orientation of a graph G to handle star edge coloring and present two more upper bounds.

Definition 3.1. Let G be a graph on n vertices with maximum degree Δ, and $p, q \le \Delta$ be two positive integers. A *well-ordered* (p, q)*-star orientation* $(\mathcal{V}, D)$ of G is a vertex enumeration $\mathcal{V} = (v_1, v_2, \ldots, v_n)$ together with the orientation D such that, for each $i \in [n]$,
(a) $d_D^+(v_i) = |E_D^+(v_i)| \le p$;
(b) for any $uv_i \in E_D^-(v_i)$, $|E_{G_i}(u)| \le q$, where G_i is the subgraph of G induced by $\cup_{j=1}^{i} E_D^-(v_j)$.

We also need to modify the definition of local ordering of G (see Definition 2.4) for digraphs.

Definition 3.2. Let G be a graph and D be an orientation of G. Let $\sigma(v)$ be a cyclic ordering of the edges in $E_D^-(v)$ for each vertex v. σ is called a local ordering of D. The distance from edge uv to wv at v with respect to σ, denoted by $\rho_{\sigma,v}(uv, wv)$, is their distance in $\sigma(v)$.

Theorem 3.3. *Let G be a graph with maximum degree Δ and let $p, q \le \Delta$ be two positive integers. Assume that G has a well-ordered (p, q)-star orientation $(\mathcal{V}, D)$. Then*

$$ch'_{st}(G) \le \begin{cases} \frac{3q+2p-1}{2}\Delta - \frac{p(q+1)}{2}, & \text{if } \Delta \le p+2; \\ \frac{3q+2p-1}{2}\Delta - \frac{p(q+3)}{2}, & \text{if } \Delta \ge p+3. \end{cases}$$

Proof. Let σ be a local ordering of D. We will define a coloring of G recursively by coloring G_1, G_2 until G_n such that the coloring of G_i is indeed a star edge coloring of G_i for each $i \in [n]$. For a given edge $uv \in E_D^-(v)$, denote

$$F_v(uv) = \{wv \in E_D^-(v) : \rho_v(uv, wv) \le \left\lfloor \frac{d_D^-(v)}{2} \right\rfloor\} \text{ and } g_v(uv) = wv \text{ where } \rho_v(uv, wv) = 1.$$

First, we color G_1 with a proper edge coloring. Note that $E(G_1)$ induces a star (possible empty).

Now we assume that G_{i-1} is already colored with an edge coloring c. We are to extend the coloring c to the edges in $E_D^-(v_i)$ to obtain a star edge coloring of G_i. Denote $\sigma(v_i) = \{u_1v_i, u_2v_i, \ldots, u_{d_D^-(v_i)}v_i\}$. Suppose that all the edges $u_1v_i, \ldots u_{j-1}v_i$ are colored and we are to color the edge u_jv_i according to the following rules (see Fig. 3).

(i) *$c(u_jv_i) \ne c(u_tv_i)$ for any $t \in [d_D^-(v_i)]$ with $t \le j-1$;*
(ii) *$c(u_jv_i) \notin c(y)$ for any $yu_j \in E_{G_i}(u_j)$ with $y \ne v_i$;*
(iii) *$c(u_jv_i) \notin c(z)$ for any $zv_i \in F_{v_i}(u_jv_i)$;*
(iv-a) *$c(u_jv_i) \notin c(x)$ for any $v_ix \in E_D^+(v_i)$ with $c(v_ix) \in c(u_j)$ or $d_{G_i}(x) \le \Delta - 1$;*
(iv-b) *for any $v_ix \in E_D^+(v_i)$ with $c(v_ix) \notin c(u_j)$ and $d_{G_i}(x) = \Delta$,*

$$c(u_jv_i) \notin \begin{cases} c(x) \setminus c(g_x(v_ix)), & \text{if } \Delta \ge p+3; \\ c(x), & \text{if } \Delta \le p+2. \end{cases}$$

Now we estimate the number of forbidden colors for u_jv_i.

(a) By (i) u_jv_i and u_kv_i should be colored with different colors for any $k \neq j$, and this requires at most $d_D^-(v_i) - 1$ forbidden colors.

(b) The number of forbidden colors from (ii) is at most $(q-1)\Delta$, since $|E_{G_{i-1}}(u_j)| = |E_{G_i}(u_j)| - 1 \le q - 1$.

(c) For each z with $zv_i \in F_{v_i}(u_jv_i)$, $|c(z)| \le |E_{G_i}(z)| \le q$ and the color $c(zv_i)$ is already counted as a forbidden color in (a). Thus the number of forbidden colors from (iii) not counted in (a) is at most $(q-1)\lfloor \frac{d_D^-(v_i)}{2} \rfloor$.

Let $a = |\{v_ix \in E_D^+(v_i) : c(v_ix) \in c(u_j) \text{ or } d_{G_i}(x) \le \Delta - 1\}|$. Then $0 \le a \le d_D^+(v_i)$.

(d) If $c(v_ix) \in c(u_j)$, then $c(v_ix)$ is counted in (b). Thus the number of forbidden colors from (iv-a) not counted in (b) is at most $a(\Delta - 1)$, and the number of forbidden colors from (iv-b) is at most $(d_D^+(v_i) - a)\Delta$ (when $\Delta \le p+2$) or $(d_D^+(v_i) - a)(\Delta - 1)$ (when $\Delta \ge p+3$). Hence the number of forbidden colors from (iv-a) and (iv-b) is at most $d_D^+(v_i)\Delta$ (when $\Delta \le p+2$) or $d_D^+(v_i)(\Delta - 1)$ (when $\Delta \ge p+3$).

Therefore, when $\Delta \ge p+3$, the total number of forbidden colors for u_jv_i is at most

$$\begin{aligned}
&(q-1)\Delta + (q-1)\left\lfloor \frac{d_D^-(v_i)}{2} \right\rfloor + d_D^+(v_i)(\Delta - 1) + d_D^-(v_i) - 1 \\
&= (q-1)\Delta + (q-1)\left\lfloor \frac{d_D^-(v_i)}{2} \right\rfloor + d_D^+(v_i)(\Delta - 2) + (d_D^+(v_i) + d_D^-(v_i)) - 1 \\
&\le (q-1)\Delta + (q-1)\left(\frac{\Delta - d_D^+(v_i)}{2} \right) + d_D^+(v_i)(\Delta - 2) + \Delta - 1 \text{ (since } d_D^+(v_i) + d_D^-(v_i) \le \Delta) \\
&= \frac{3q-1}{2}\Delta + \left(\frac{2\Delta - 3 - q}{2} \right) d_D^+(v_i) - 1 \\
&\le \frac{3q-1}{2}\Delta + \left(\frac{2\Delta - 3 - q}{2} \right) p - 1 \quad \text{(since } d_D^+(v_i) \le p) \\
&= \frac{3q+2p-1}{2}\Delta - \frac{p(q+3)}{2} - 1.
\end{aligned}$$

If $\Delta \le p+2$, then similar calculation yields that the number of forbidden colors is at most $\frac{3q+2p-1}{2}\Delta - \frac{p(q+1)}{2} - 1$.

Therefore, $\frac{3q+2p-1}{2}\Delta - \frac{p(q+1)}{2}$ colors (when $\Delta \le p+2$) or $\frac{3q+2p-1}{2}\Delta - \frac{p(q+3)}{2}$ colors (when $\Delta \ge p+3$) are enough to complete the coloring process.

Finally we show that this coloring is indeed a star edge coloring. It suffices to show, in the graph G_i, for each $j \in [d_D^-(v_i)]$, after coloring u_jv_i, it does not produce a bicolored path or cycle of length four. Suppose to the contrary that there is a bicolored path or cycle P of length four containing the edge u_jv_i. Obviously by (ii), P is not a cycle and v_i is not an endpoint of P. Let u_jv_ix be a subpath in P. Then either $c(u_jv_i) \in c(x)$ or $c(v_ix) \in c(u_j)$.

If $xv_i \in E_D^-(v_i)$, then $x = u_k$ for some $k \in [d_D^-(v_i)]$. By (ii), $c(u_jv_i) \notin c(y)$ for any $yu_j \in E_{G_i}(u_j)$ with $y \neq v_i$, and so u_k is not an endpoint of P. Similarly, u_j is not an endpoint of P. This implies $c(u_jv_i) \in c(u_k)$ and $c(u_kv_i) \in c(u_j)$. By (iii), $u_kv_i \notin F_{v_i}(u_jv_i)$ and $u_jv_i \notin F_{v_i}(u_kv_i)$. Thus $\rho_{v_i}(u_jv_i, u_kv_i) \ge \lfloor \frac{d_D^-(v_i)}{2} \rfloor + 1$ and $\rho_{v_i}(u_kv_i, u_jv_i) \ge \lfloor \frac{d_D^-(v_i)}{2} \rfloor + 1$. Therefore we can obtain the following contradiction:

$$d_D^-(v_i) = \rho_{v_i}(u_jv_i, u_kv_i) + \rho_{v_i}(u_kv_i, u_jv_i) \ge 2\left\lfloor \frac{d_D^-(v_i)}{2} \right\rfloor + 2 \ge d_D^-(v_i) + 1.$$

Now we assume $v_ix \in E_D^+(v_i)$. By (ii) again, x is not an endpoint of P which implies $c(u_jv_i) \in c(x)$. By (iv-a) and (iv-b), $c(v_ix) \notin c(u_j)$ and $d_{G_i}(x) = \Delta \ge p+3$. Thus u_j is an endpoint of P. Let $P = u_jv_ixx_1x_2$. By (iv-b), $x_1x \in E_D^-(x)$ and $xx_1 = g_x(v_ix)$ (meaning $\rho_x(v_ix, x_1x) = 1$).

Since P is bicolored, we have $c(xx_1) = c(u_jv_i)$, and so $c(u_jv_i) = c(xx_1) = c(g_x(v_ix))$ and $d_{G_{i-1}}(x) = d_{G_i}(x) = \Delta \ge p+3$ by (iv-a) and (iv-b). Hence $d_D^-(x) \ge \Delta - p \ge 3$. Note $c(v_ix) \in c(x_1)$.

If x_1x_2 is colored before v_ix, then $x_1x \notin F_x(v_ix)$ by (iii). But we have $d_D^-(x) \ge 3$ and $1 = \rho_x(v_ix, x_1x) \le \lfloor \frac{d_D^-(x)}{2} \rfloor$, which implies $x_1x \in F_x(v_ix)$ by definition, a contradiction.

Now assume that x_1x_2 is colored after v_ix. By (ii), x_1x_2 is oriented from x_2 to x_1 since $c(x_1x_2) \in c(x)$. By (iv-a) and (iv-b), we have $c(x_1x_2) = c(g_x(x_1x))$ which implies $\rho_x(x_1x, v_ix) = 1$. Thus we obtain the following contradiction:

$$3 \le d_D^-(x) = \rho_x(v_ix, x_1x) + \rho_x(x_1x, v_ix) = 2.$$

Therefore c is a star edge coloring and thus completes the proof of the theorem. ∎

By modifying the coloring algorithm in the proof of Theorem 3.3, we also obtain another upper bound for $ch'_{st}(G)$ for any graph G with a well-ordered (p,q)-star orientation.

Theorem 3.4. *Let G be a graph with a well-ordered (p,q)-star orientation $(\mathcal{V}, D)$. Let $\Delta \ge 3$ be the maximum degree of G and let $q \ge 2$. Then*

$$ch'_{st}(G) \le (p+q)\Delta + q^2 - 3q - p + 2.$$

Proof. We adopt the same notation as in Theorem 3.3 and modify the coloring rules as below.

We assume that G_{i-1} is already colored with an edge coloring c and we extend the coloring c to the edges in $E_D^-(v_i)$ to obtain a star edge coloring of G_i. Assume that all the edges $u_1v_i, \ldots u_{j-1}v_i$ are colored. We are to color the edge u_jv_i according to the following rules.

(i) $c(u_jv_i) \neq c(u_tv_i)$ *for each* $t \leq j-1$;
(ii) $c(u_jv_i) \notin c(y)$ *for any* $yu_j \in E_{G_i}(u_j)$ *with* $y \neq v_i$;
(iii) $c(u_jv_i) \notin c(z)$ *for any* $zv_i \in E_D^-(v_i)$ *with* $c(zv_i) \in c(u_j)$;
(iv) $c(u_jv_i) \notin c(x)$ *for any* $v_ix \in E_D^+(v_i)$.

Denote $b = |c(E_D^-(v_i)) \cap c(u_j)|$. Then $b \leq q-1$. Similar to Theorem 3.3, the total number of forbidden colors for u_jv_i is at most

$$\begin{aligned}
&(q-1)\Delta + (q-1)b + d_D^+(v_i)\Delta + (d_D^-(v_i) - b - 1)\\
&= (q-1)\Delta + (q-2)b + d_D^+(v_i)(\Delta - 1) + (d_D^+(v_i) + d_D^-(v_i)) - 1\\
&\leq (q-1)\Delta + (q-2)(q-1) + p(\Delta-1) + \Delta - 1\\
&= (p+q)\Delta + q^2 - 3q - p + 1.
\end{aligned}$$

Since there are $(p+q)\Delta + q^2 - 3q - p + 2$ colors, one can always find a color for u_jv_i.

Now we show that after coloring u_jv_i, the new coloring is a star edge coloring. Suppose to the contrary that P is a bicolored path or cycle of length four containing the edge u_jv_i. By (ii), P is not a cycle and v_i is not an endpoint of P. Let u_jv_ix be a subpath of P. By (ii) again, x is not an endpoint of P. Thus $c(u_jv_i) \in c(x)$, and so $xv_i \in E_D^-(v_i)$ by (iv). Thus by (iii), $c(xv_i) \notin c(u_j)$, which implies that u_j is an endpoint of P. Denote $P = u_jv_ixx_1x_2$ where $u_jv_i, xv_i \in E_D^-(v_i)$ and $c(xv_i) = c(x_1x_2) \in c(x_1)$. Thus xx_1 and x_1x_2 both are colored before xv_i. By (ii), $c(xv_i) \notin c(x_1)$, a contradiction. This proves Theorem 3.4. ■

We shall show that every k-degenerate graph admits a well-ordered (k,k)-star orientation, and then apply Theorems 3.3 and 3.4 to obtain upper bounds on list star edge chromatic index of k-degenerate graphs, which will prove Theorem 1.9(a) and (b).

Lemma 3.5. *Every k-degenerate graph admits a well-ordered (k,k)-star orientation.*

Proof. Let G be a k-degenerate graph. We shall find $G_n, G_{n-1}, \ldots, G_1$ and $v_n, \ldots, v_1$ recursively. Define $G_n = G$. We assume G_i is determined and we are to find v_i and G_{i-1} according to the following.

(A1) *If $V_{\geq k+1}(G_i) \neq \emptyset$, choose v_i to be a vertex in $V_{\geq k+1}(G_i)$ whose degree is at most k in the subgraph $G_i[V_{\geq k+1}(G_i)]$ of G_i induced by $V_{\geq k+1}(G_i)$.*

(A2) *If $V_{\geq k+1}(G_i) = \emptyset$ and $E(G_i) \neq \emptyset$, choose v_i to be a vertex with maximum degree in G_i.*

(A3) *If $V_{\geq k+1}(G_i) = \emptyset$ and $E(G_i) = \emptyset$, let v_i be any vertex in $V(G) \setminus \{v_n, \ldots, v_{i+1}\}$.*

(B) *For each edge $uv_i \in E(G_i)$ with $|E_{G_i}(u)| \leq k$, orient the edge uv_i from u to v_i.*

(C) *Set $G_{i-1} = G_i - \{uv_i \in E(G_i) : |E_{G_i}(u)| \leq k\}$.*

Note that, in (A1) such a vertex exists since G is k-degenerate graph and $G_i[V_{\geq k+1}(G_i)]$ is a subgraph of G. We claim that this defines a proper vertex enumeration $\mathcal{V} = (v_1, v_2, \ldots, v_n)$. To this end, we show that $v_i \neq v_j$ for any $i \neq j$. Suppose to the contrary that a vertex v is labeled with v_i and v_j for some $i > j$.

We first claim that the degree of v_j in G_j is not zero. Otherwise, $E(G_j) = \emptyset$ by (A2) and (A3), and by (A3) again, v_j is not selected, a contradiction. Thus $v_i \in V_{\geq k+1}(G_i)$ otherwise the degree of v_i is zero in G_t for any $t = i-1, \ldots, j$ by (C). By (C), for every w such that $v_iw \in E_{G_i}(v_i) \setminus E_D^-(v_i)$, $|E_{G_{i-1}}(w)| \geq k+1$. Furthermore, by (A1) and (C), we have $E_{G_{i-1}}(v_i) \leq k$. Thus according to (A1), w is always chosen before vertex v_j for every w such that $v_iw \in E_D^-(w)$. This implies that v_j has degree zero in G_j, a contradiction. This proves $v_i \neq v_j$ for any $i \neq j$.

Clearly, this defines an orientation D satisfying (a) and (b) in Definition 3.1. Therefore, $(\mathcal{V}, D)$ is a well-ordered (k,k)-star orientation with $\mathcal{V} = (v_1, v_2, \ldots, v_n)$. ■

Proof of Theorem 1.9 (a) and (b). Theorem 1.9(a) with $\Delta \geq k+3$ and Theorem 1.9(b) are implied by Theorems 3.3 and 3.4 with $p = q = k$, together with Lemma 3.5. It remains to show Theorem 1.9(a) when $\Delta \in \{k, k+1, k+2\}$. We may also assume $\Delta \geq 4$ as the case of $\Delta = 2$ is trivial and the case of $\Delta = 3$ follows from Theorem 1.7 (see also [12]).

We compare the bounds in Theorem 2.6 with the desired bound $\frac{5k-1}{2}\Delta - \frac{k(k+3)}{2}$ in all cases. The bounds in Theorem 2.6 are better when $\Delta \in \{k, k+1\}$. For the case of $\Delta = k+2$, when Δ is odd we have $k \geq 3$ and

$$\left(\frac{5k-1}{2}\Delta - \frac{k(k+3)}{2}\right) - \left(\Delta^2 + \frac{k-5}{2}\Delta + \frac{3k+3}{2}\right) = \frac{1}{2}k^2 - k - \frac{3}{2} \geq 0;$$

when Δ is even and $k \geq 4$, we have

$$\left(\frac{5k-1}{2}\Delta - \frac{k(k+3)}{2}\right) - \left(\Delta^2 + \frac{k-4}{2}\Delta + 2k - 1\right) = \frac{1}{2}k^2 - 2k \geq 0.$$

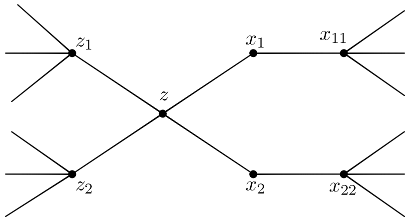

Fig. 4. A possible configuration.

Now it remains to verify the final case that $\Delta = 4$ and $k = 2$. That is, we will show the following statement.

Every 2-degenerate graph G with maximum degree 4 is star 13-edge-choosable.

Let G together with an edge 13-list L be a counterexample to the above statement with $|E(G)|$ minimized.

Let $xy \in E(G)$. By the minimality of G, $G - xy$ has a list star edge coloring c with $c(e) \in L(e)$ for each $e \in E(G) \setminus \{xy\}$. Denote $A(xy) = \bigcup_{w \in N(x) \cup N(y) \setminus \{x,y\}} c(w)$.

(I) *For any* $xy \in E(G)$, $|A(xy)| \geq 13$ *and thus* $\delta(G) = 2$.

Otherwise, $L(xy) \setminus A(xy) \neq \emptyset$. Thus one can always pick a color in $L(xy) \setminus A(xy)$ to color xy to extend c to be a list star edge coloring of G, a contradiction.

Let z be a vertex with minimum degree in $G[V_{\geq 3}]$. Then z has a neighbor x_1 of degree 2 in G since G is 2-degenerate and $\delta(G) = 2$.

(II) $d_G(z) = 4$ *and* z *has exactly two neighbors of degree* 2.

If $d_G(z) = 3$, then $|A(zx_1)| \leq 4 + 4 + 4 = 12 < 13$, a contradiction to (I).

If z has at least three neighbors of degree 2, then $|A(zx_1)| \leq 4 + 4 + 2 + 2 = 12 < 13$, a contradiction to (I) again.

By (II), let x_1 and x_2 be the two neighbors of z with degree 2 and z_1, z_2 be the other two neighbors of z. Let $x_{ii} \neq z$ be the other neighbor of x_i for each $i = 1, 2$ (see Fig. 4).

By the minimality of G, let c' be a star edge coloring of $G - x_1 - x_2$. We are to extend c' to a star edge coloring c of G below. Since $|\bigcup_{x \in N(x_{ii}) \setminus \{x_i\}} c'(x)| \leq 12$ and $|L(x_i x_{ii})| = 13$ for each $i = 1, 2$, we first color $x_i x_{ii}$ with a color in $L(x_i x_{ii}) \setminus \bigcup_{x \in N(x_{ii}) \setminus \{x_i\}} c'(x)$. Denote c to be the new coloring of $G - zx_1 - zx_2$ after coloring $x_1 x_{11}$ and $x_2 x_{22}$.

(III) $c(x_i x_{ii}) \notin c(z)$ *for each* $i = 1, 2$.

Without loss of generality, assume that $c(x_1 x_{11}) \in c(z)$. Then $|c(z_1) \cup c(z_2) \cup c(x_{11})| \leq 11$, and we first color zx_1 with a color α such that $\alpha \in L(zx_1) \setminus [c(z_1) \cup c(z_2) \cup c(x_{11})]$ and $\alpha \neq c(x_2 x_{22})$. Clearly, this coloring of $G - zx_2$ is a star edge coloring of $G - zx_2$. If $c(x_2 x_{22}) \in c(z)$, then $|A(zx_2)| \leq 12$ since $c(x_1 x_{11}) \in c(z)$, a contradiction to (I). Thus, $c(x_2 x_{22}) \notin c(z)$.

Since $c(x_1 x_{11}) \in c(z)$, $|c(z_1) \cup c(z_2) \cup c(x_1) \cup \{c(x_2 x_{22})\}| \leq 10$, and so we color zx_2 with a color $\beta \in L(zx_2) \setminus [c(z_1) \cup c(z_2) \cup c(x_1) \cup \{c(x_2 x_{22})\}]$.

We verify that this results in a star edge coloring. Suppose that P is a bicolored path (or cycle) of length four containing zx_2. By the coloring of zx_2, we have $|P \cap E(t)| \leq 1$ for each $t \in \{z_1, z_2, x_1\}$, and so $|P \cap E(x_{22})| = 2$ and z is an endpoint of P since $c(x_2 x_{22}) \notin c(z)$. However $c(x_2 x_{22}) \notin c(w)$ for each $w \in N(x_{22})$ and $w \neq x_2$. This implies that the length of P is at most three and thus proves (III).

The final step: By (III), we may assume $c(x_i x_{ii}) \notin c(z)$ for each $i = 1, 2$. Since $|c(z_1) \cup c(z_2) \cup \{c(x_1 x_{11}), c(x_2 x_{22})\}| \leq 10$, one can color the edges zx_1, zx_2 properly such that $c(zx_i) \in L(zx_i) \setminus [c(z_1) \cup c(z_2) \cup \{c(x_1 x_{11}), c(x_2 x_{22})\}]$ for each $i = 1, 2$.

It remains to check this is a star edge coloring. Suppose that P is a bicolored path or cycle of length four containing zx_1 or zx_2. Without loss of generality, assume that P contains zx_1. For each $i = 1, 2$, z_i is not an endpoint of P since $c(x_1 x_{11}) \notin c(z)$ and z_i is not contained in P either since $c(zx_1) \notin c(z_i)$ for each $i = 1, 2$.

Since $c(x_1 x_{11}) \notin \bigcup_{x \in N(x_{11}) \setminus \{x_1\}} c(x)$, z is not an endpoint of P. Thus P contains $x_1 z x_2$. Since $|E(P)| = 4$, either $c(zx_1) = c(x_2 x_{22})$ or $c(zx_2) = c(x_1 x_{11})$. However by the choice of $c(zx_i)$, $c(zx_i) \notin \{c(x_1 x_{11}), c(x_2 x_{22})\}$ for each $i = 1, 2$. This contradiction proves that c is a star edge coloring of G and thus completes the proof. ∎

Acknowledgments

The authors would like to thank the anonymous referees for their helpful comments, which substantially improves the presentation of this paper. The first author was partially supported by the Talent Fund Project of Tianjin Normal University, China (5RL159). The second author was partially supported by the Fundamental Research Funds for the Central Universities. The fourth author was partially supported by NSF-China grant: NSFC 11171288.

References

[1] L. Bezegová, B. Lužar, M. Mockovčiaková, R. Soták, R. Škrekovski, Star edge coloring of some classes of graphs, J. Graph Theory 81 (1) (2016) 73–82.

[2] K. Deng, X.S. Liu, S.L. Tian, Star edge coloring of trees, J. Shandong Univ. Nat. Sci. 46 (8) (2011) 84–88, (in Chinese).

[3] Z. Dvořák, B. Mohar, R. Šámal, Star chromatic index, J. Graph Theory 72 (2013) 313–326.

[4] G. Fertin, A. Raspaud, B. Reed, On star coloring of graphs, in: Lecture Notes 483 in Comput. Sci. 2204, Springer, Berlin, 2001, pp. 140–153.
[5] B. Grünbaum, Acyclic colorings of planar graphs, Israel J. Math. 14 (1973) 390–408.
[6] S. Kerdjoudj, A. Kostochka, A. Raspaud, List star edge coloring of subcubic graphs, Discuss. Math. Graph Theory 38 (2018) 1037–1054.
[7] S. Kerdjoudj, K. Pradeep, A. Raspaud, List star chromatic index of sparse graphs, Discrete Math. 341 (2018) 1835–1849.
[8] S. Kerdjoudj, A. Raspaud, List star edge coloring of sparse graphs, Discrete Appl. Math. 238 (2018) 115–125.
[9] H. Lei, Y. Shi, Z. Song, Star chromatic index of subcubic multigraph, J. Graph Theory 88 (2018) 566–576.
[10] H. Lei, Y. Shi, Z. Song, T. Wang, Star 5-edge-colorings of subcubic multigraphs, Discrete Math. 88 (2018) 566–576.
[11] X.S. Liu, K. Deng, An upper bound on the star chromatic index of graphs with $\Delta \geq 7$, J. Lanzhou Univ. Nat. Sci. 2 (2008) 98–99.
[12] B. Lužar, M. Mockovčiaková, R. Soták, Note on list star edge-coloring of subcubic graphs, J. Graph Theory 90 (2019) 304–310.
[13] Y. Wang, W. Wang, Y. Wang, Edge-partition and star chromatic index, Appl. Math. Comput. 333 (2018) 480–489.