Packing spanning trees in highly essentially connected graphs

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ABSTRACT

Let $\tau(G)$ be the maximum number of mutually edge-disjoint spanning trees contained in a graph G and let $\kappa'(G)$ denote the edge-connectivity of G. As a corollary of the spanning trees packing theorem by Nash-Williams and Tutte, it is known that if $\kappa'(G) \geq 2k$, then $\tau(G) \geq k$. An edge-cut X of G is an essential edge-cut if $G-X$ contains at least two nontrivial components; and G is essentially k-edge-connected if G does not have an essential edge-cut of size less than k. In this paper, we prove that every g-edge-connected, essentially h-edge-connected graph G with $g \geq k + 1$ and $h \geq \frac{g^2}{g-k} - 2$ satisfies $\tau(G) \geq k$. This result is sharp in the sense that there exist infinitely many graphs showing that neither inequality in the hypothesis can be relaxed. Applications to circular flows of graphs, spanning connectivity of line graphs and supereulerian width of graphs are discussed. In particular, we obtained the following, for given integers g and k with $k > 1$ and $2k - 1 \geq g \geq k + 1$.

(i) Every 5-edge-connected essentially 23-edge-connected graph admits a nowhere-zero 3-flow.

(ii) Every 7-edge-connected essentially 47-edge-connected graph has circular flow number less than 3.

(iii) Every 8-edge-connected essentially 20-edge-connected planar graph has circular 5/2-flow.

(iv) Every g-edge-connected essentially ($\lceil \frac{g^2}{g-k} \rceil - 2$)-edge-connected graph has supereulerian width at least $k + 1$.

(v) For a line graph $G = L(H)$, if G is ($\lceil \frac{g^2}{g-k} \rceil - 2$)-connected and $\delta(H) \geq g$, then G is spanning k-connected.

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1. Introduction

In this paper, graphs are finite and loopless, but may contain parallel edges. We follow [4] for undefined terminologies and notation. Let $\tau(G)$ be the maximum number of mutually edge-disjoint spanning trees contained in a graph G, and let $\kappa'(G)$ and $\Delta(G)$ denote the edge-connectivity and the maximum degree of G, respectively. For an edge subset $E' \subseteq E(G)$, define the contraction G/E' to be the graph obtained from G by identifying the two end vertices of each edge in E' and then deleting the resulting loops. If H is a subgraph of G, we often use G/H for $G/E(H)$. As a widely used application of Nash-Williams and Tutte's theorem [33,37] on spanning tree packing, it is known that

$$\tau(G) \geq \lfloor \frac{\kappa'(G)}{2} \rfloor,$$

as shown by Polesskiĭ [32], Kundu [24] and Catlin [7], among others. We restate it as follows.

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Fig. 1. Some graphs with no essential edge-cut.

Theorem 1.1 ([33,37]). *Every* 2*k-edge-connected graph contains k edge-disjoint spanning trees.*

Examples like K_{2k} indicated that there exist graphs G with $\kappa'(G) = 2k - 1$ and $\tau(G) \leq k - 1$. It is natural to seek conditions on a graph G with $\kappa'(G) < 2k$ which can warrant $\tau(G) \geq k$. The main goal of this paper is to address this problem.

A graph is nontrivial if it contains at least one nonloop edge. An edge-cut X of a connected graph G is **essential** if at least two components of $G - X$ are nontrivial. A graph is **essentially** k**-edge-connected** if it does not have an essential edge-cut with fewer than k edges. It is easy to observe that a loopless connected graph G on $n = |V(G)| \geq 2$ vertices does not have an essential edge-cut if and only if either G is spanned by a K_3 or G has a vertex v_0 such that $E(G - v_0) = \emptyset$ (i.e. the underlying simple graph of G is a star $K_{1,n-1}$), see Fig. 1. For a graph G which is spanned by a K_3, we define the essential edge connectivity of G to be $\Delta(G)$; and if G has a vertex v_0 with $E(G - v_0) = \emptyset$, then define the essential edge connectivity of G to be infinity. As every edge cut of a contraction of G is also an edge cut of G, it follows that the edge connectivity, and the essential edge connectivity are preserved under contraction.

Chartrand and Stewart [8] first introduced the concept of essential edge connectivity as they observed that if $L(G)$ is not complete, then $L(G)$ is k-connected if and only if G is essentially k-edge-connected. In the study of Hamiltonian line graph, Zhan's argument in [42] actually showed that *every* 3*-edge-connected essentially* 7*-edge-connected graph contains two edge-disjoint spanning trees,* which in turn results that every 7-connected line graph is hamiltonian-connected (see [42] or Section 3 for details). The notion of essential edge connectivity is also known as restricted edge connectivity in the literature, as proposed by Esfahanian in [12].

The main result of this paper is the following essential edge connectivity version of Theorem 1.1, which provides a sufficient condition for spanning tree packing.

Theorem 1.2. *Let* k, g, h *be positive integers such that* $k + 1 \leq g \leq 2k - 1$ *and* $h \geq \frac{g^2}{g-k} - 2$. *Then every* g*-edge-connected essentially* h*-edge-connected graph contains* k *edge-disjoint spanning trees.*

We remark that the essential edge connectivity condition in Theorem 1.2 is tight as can be seen in Propositions 2.2 and 2.3.

Theorem 1.2 can be applied to circular flows of graphs, and to spanning connectivity of line graphs and to supereulerian width problem of graphs. In the next section, we present the proof of Theorem 1.2. Applications of Theorem 1.2 to circular flows, spanning connectivity of line graphs and to the supereulerian width of graphs will be discussed in Section 3. Our concluding remarks are presented in Section 4.

2. Essential edge connectivity and spanning tree packing

Throughout this section, i and k denote two nonnegative integers. For a graph G, define $D_i(G) = \{v \in V(G) : d_G(v) = i\}$, $d_i(G) = |D_i(G)|$, $D_{\leq k}(G) = \cup_{i \leq k} D_i(G)$, and $D_{\geq k}(G) = \cup_{i \geq k} D_i(G)$. When the graph G is understood from the context, we often use $D_i, d_i, D_{\geq k}$ and $D_{\leq k}$ for $D_i(G), d_i(G), D_{\geq k}(G)$ and $D_{\leq k}(G)$, respectively. For vertex subsets $U, W \subseteq V(G)$, let $[U, W]_G = \{uw \in E(G) : u \in U, w \in W\}$, and we use $[S, V(G) - S]_G$ to denote an edge-cut of G. We use $E_G(v) = [\{v\}, V(G) - \{v\}]$ to denote a trivial edge-cut for convenience. For any edge $e = uv \in E(G)$, we define $d_G(e) = d_G(u) + d_G(v) - 2$, called the degree of e in G; and $\xi(G) = \min_{e \in E(G)} d_G(e)$, called the **minimum edge degree** of G. The subscript G may be omitted when G is understood from the context. The next theorem will be useful.

Theorem 2.1 (Li et al. [28] , Xu et al. [39]). *Every edge-transitive simple graph which is not the star graph has essential edge connectivity equal its minimum edge degree.*

2.1. Tightness of Theorem 1.2

We start with two examples, which would indicate that the conditions $k+1 \leq g \leq 2k-1$ and $h \geq \frac{g^2}{g-k} - 2$ in Theorem 1.2 are tight.

Proposition 2.2. *For any integer* $k \geq 2$, *and for any sufficiently large integer* $\ell > 0$, *there exists a* k-edge-connected, essentially ℓ-edge-connected graph G with $\tau(G) < k$.

Proof. To see that, we choose $\ell > 2k - 2$. It is easy to observe (or use Theorem 2.1 to see) that the complete bipartite graph $K_{k,\ell-k+2}$ is k-edge-connected and essentially ℓ-edge-connected. As $|E(K_{k,\ell-k+2})| = k(\ell - k + 2) < k(|V(K_{k,\ell-k+2})| - 1)$, we conclude that $\tau(K_{k,\ell-k+2}) < k$. Infinitely many such examples can be obtained as follows. Let G_i be a copy of $K_{k,\ell-k+2}$ with a distinguished edge $u_i v_i$ for each $i = 1, 2, \ldots, n, n+1$, where $u_1, u_2, \ldots, u_n$ and v_{n+1} are degree k vertices, and $v_1, v_2, \ldots, v_n$ and u_{n+1} are degree $\ell - k + 2$ vertices. Obtain a new graph G by identifying $u_1, \ldots, u_{n+1}$ to form a new vertex u, identifying $v_1, \ldots, v_{n+1}$ to form a new vertex v, and deleting the edges parallel to uv. Then it is routine to verify that the graph G is still k-edge-connected and essentially ℓ-edge-connected. Since $|V(G)| = (n+1)\ell + 2$ and $|E(G)| = k(\ell - k + 2)(n+1) - n < k|V(G)| - k$, we have $\tau(G) < k$. As n is an arbitrary positive integer, this generates infinitely many such examples. $\square$

Proposition 2.3. *For given integers g and k with $3 \leq k + 1 \leq g \leq 2k - 1$, there exists a g-edge-connected essentially $\left(\lceil \frac{g^2}{g-k} \rceil - 3 \right)$-edge-connected graph G with $\tau(G) < k$.*

Proof. We will construct such a graph and our construction is motivated by the method used in [14].

For integers $g \geq 3$ and $t > 1$, let S be a set with $|S| = g + t - 1$, $X = S^{[g]}$ be the collection of all g-subsets of S and $Y = S^{[g-1]}$ be the collection of all $(g-1)$-subsets of S. Define $G(g,t)$ to be the bipartite graph with vertex bipartition (X, Y), where $x \in X$ is adjacent to $y \in Y$ if and only if, as subsets of S, $y \subset x$. Then every $x \in X$ has degree g in $G(g,t)$ and every $y \in Y$ has degree t in $G(g,t)$, and so $|E(G(g,t))| = g\binom{g+t-1}{g}$. Moreover, $G(g,t)$ is edge-transitive (see [14]), and so by Theorem 2.1, $G(g,t)$ is g-edge-connected essentially $(g + t - 2)$-edge-connected.

Choose $t = \lceil \frac{g^2}{g-k} \rceil - g - 1$ and let $h_1 = \lceil \frac{g^2}{g-k} \rceil - 3$. Then $g + t - 2 = h_1 = \lceil \frac{g^2}{g-k} \rceil - 3 \geq \frac{g^2}{g-k} - 3$. Algebraic manipulation yields that $\frac{kg-1}{g-k} \geq t \geq \frac{kg}{g-k} - 1$. Set $s = \frac{1}{t}\binom{g+t-1}{g}$. Then $|X| = ts$. As $t \geq \frac{kg}{g-k} - 1$ and $2k - 1 \geq g \geq k + 1 \geq 3$, we have

$$
\begin{aligned}
s - 2k &= \frac{1}{t} \cdot \frac{(g+t-1)\cdots(t+1)t}{g!} - 2k \geq \frac{(g+t-1)(t+1)}{2g} - 2k = \frac{t^2 + gt + g - 1 - 4gk}{2g} \\
&\geq \frac{(\frac{kg}{g-k} - 1)^2 + g(\frac{kg}{g-k} - 1) + g - 1 - 4gk}{2g} = \frac{k}{2(g-k)^2}[-3g^2 + (8k-2)g + 2k - 4k^2] \\
&\geq \frac{k}{2(g-k)^2}[-3(2k-1)^2 + (8k-2)(2k-1) + 2k - 4k^2] = \frac{k(2k-1)}{2(g-k)^2} > 0.
\end{aligned}
$$

It follows from $s > 2k$ and $t \leq \frac{kg-1}{g-k}$ that

$$
\begin{aligned}
k(|X| + |Y| - 1) - |E(G(g,t))| &= k(ts + gs - 1) - gts = (k-g)st + kgs - k \\
&\geq s(k-g)\frac{kg-1}{g-k} + kgs - k = s - k > k.
\end{aligned}
$$

Therefore, $|E(G(g,t))| < k(|V(G(g,t))| - 1)$, implying that $\tau(G(g,t)) < k$. An infinite family of such examples may be obtained with a similar construction as in Proposition 2.2. For $i = 1, 2, \ldots, n+1$, let H_i be a copy of $G(g,t)$ with a distinguished edge $u_i v_i$, where $u_1, \ldots, u_n$ and v_{n+1} are degree g vertices, and $v_1, \ldots, v_n$ and u_{n+1} are degree t vertices. Then we identify v_i's to become a new vertex v, and identify u_i's to become a new vertex u, and then delete the parallel edges to obtain the resulting graph H. Note that H is g-edge-connected essentially $(g + t - 2)$-edge-connected. In addition, we have

$$
\begin{aligned}
&k(|V(H)| - 1) - |E(H)| \\
&= k[(|X| + |Y| - 2)(n+1) + 1] - [(n+1)|E(G(g,t))| - n] \\
&= (n+1)[k(|X| + |Y| - 1) - |E(G(g,t))|] - kn + n \\
&> k(n+1) - kn + n > 0.
\end{aligned}
$$

Hence this implies $\tau(H) < k$. $\square$

2.2. Proof of Theorem 1.2

To complete the proof of Theorem 1.2, we need the next lemma, which follows from arguments of Nash-Williams in [34]. A detailed proof can be found in Theorem 2.4 of [40].

Lemma 2.4 (Nash-Williams [34]). *Let G be a nontrivial graph and let $k > 0$ be an integer. If $|E(G)| \geq k(|V(G)| - 1)$, then G has a nontrivial subgraph H such that $\tau(H) \geq k$.*

Proof of Theorem 1.2. By contradiction, we may assume that there exists a g-edge-connected essentially h-edge-connected graph G with $\tau(G) < k$. Choose such a counterexample G with $|E(G)|$ minimized. By definition, if $|V(G)| = 2$, then $\tau(G) = \kappa'(G)$. Hence we assume that $|V(G)| \geq 3$. If G is spanned by a K_3, then, by definition, G has a vertex of degree at least $h \geq \frac{g^2}{g-k} - 2$, and so $|E(G)| \geq \frac{1}{2}(\frac{g^2}{g-k} - 2 + 2(k+1)) \geq 2k$, implying that $\tau(G) \geq k$ by Lemma 2.4. If G has a vertex v_0

such that $E(G - v_0) = \emptyset$, then it is routine to show that $\tau(G) = \kappa'(G)$ in this case, and so we may assume that G has at least one essential edge-cut and that $|V(G)| \geq 4$.

We first claim that $|E(H)| < k(|V(H)| - 1)$ *for any nontrivial subgraph H of G*. Otherwise, by Lemma 2.4, there exists a nontrivial subgraph H with $\tau(H) \geq k$. By definition of contraction, G/H is g-edge-connected and essentially h-edge-connected. By the minimality of G, G/H has k edge-disjoint spanning trees, say $T'_1, T'_2, \ldots, T'_k$. Let $T''_1, T''_2, \ldots, T''_k$ be k edge-disjoint spanning trees of H and define $T_i = G[E(T'_i) \cup E(T''_i)]$, with $1 \leq i \leq k$. Then $T_1, T_2, \ldots, T_k$ are k edge-disjoint spanning trees in G, contrary to the choice of G. This verifies the claim.

Therefore, we must have

$$|E(G)| < k(|V(G)| - 1), \tag{1}$$

and

$$|[u, v]| < k \text{ for any } uv \in E(G). \tag{2}$$

Hence $[u, v]$ is not an edge-cut of G by (2).

We further claim that, *for any edge $uv \in E(G)$*,

$$d(u) \geq h + 2 - d(v) \geq \frac{g^2}{g-k} - d(v). \tag{3}$$

This is clear if $[\{u, v\}, V(G) - \{u, v\}]$ is an essential edge-cut of G. Assume that $[\{u, v\}, V(G) - \{u, v\}]$ is not an essential edge-cut. Then $V(G) - \{u, v\}$ is an independent set. Since G is g-edge-connected, it follows from (2) that for any $x \in V(G) - \{u, v\}$, both $xu \in E(G)$ and $xv \in E(G)$. Thus, both $[\{x, u\}, V(G) - \{x, u\}]$ and $[\{x, v\}, V(G) - \{x, v\}]$ are essential edge-cuts of G since $|V(G)| \geq 4$. Hence

$$d(u) + d(v) = |[\{x, u\}, V(G) - \{x, u\}]| + |[\{x, v\}, V(G) - \{x, v\}]| \geq 2h \geq h + 2.$$

This verifies (3).

Since $f(x) = \frac{x-2k}{x}$ is increasing over $[2k, \infty)$, it follows by (3) that

$$\sum_{e=uv \in E_G(v)} \left[\frac{d(u) - 2k}{d(u)} \right] - (2k - d(v)) \geq \sum_{e=uv \in E_G(v)} \left[1 - \frac{2k}{\frac{g^2}{g-k} - d(v)} \right] - (2k - d(v))$$

$$= 2d(v) - 2k - \frac{2kd(v)}{\frac{g^2}{g-k} - d(v)}$$

$$= \frac{-2(g-k)d^2(v) + 2g^2 d(v) - 2kg^2}{g^2 - (g-k)d(v)}. \tag{4}$$

Define $J(x) = -2(g-k)x^2 + 2g^2 x - 2kg^2$ on $[g, 2k-1]$. As the derivative $J'(x) = -4(g-k)x + 2g^2 \geq 2(g - 2k + 1)^2 + 2(2k-1) > 0$, we conclude that $J(x)$ is an increasing function on $[g, 2k-1]$, and so for any x with $g \leq x \leq 2k-1$,

$$J(x) \geq J(g) = -2(g-k)g^2 + 2g^2 g - 2kg^2 = 0. \tag{5}$$

Since $g \geq k + 1$, for any $v \in D_{\leq 2k-1}$, we have $g^2 - (g-k)d(v) > 0$, and so by (4) and (5),

$$\sum_{e=uv \in E_G(v)} \frac{d(u) - 2k}{d(u)} \geq \frac{J(d(v))}{g^2 - (g-k)d(v)} + (2k - d(v)) \geq 2k - d(v). \tag{6}$$

As $2|E(G)| = \sum_{i \geq g} id_i = \sum_{i=g}^{2k-1} id_i + \sum_{i \geq 2k} id_i$, we have

$$2|E(G)| - 2k(|V(G)| - 1) = 2k + \sum_{i=g}^{2k-1}(i - 2k)d_i + \sum_{i \geq 2k}(i - 2k)d_i$$

$$= 2k + \sum_{i \geq 2k}(i - 2k)d_i - \sum_{i=g}^{2k-1}(2k - i)d_i. \tag{7}$$

If $v \in D_{\leq 2k-1}$ and $u \in N_G(v)$, then by (3), for any $u \in N_G(v)$, $d(u) \geq \frac{g^2}{g-k} - (2k-1) = \frac{(g-2k)^2}{g-k} + 2k + 1 \geq 2k + 1$. Hence by (6), we have

$$\sum_{i=g}^{2k-1}(2k - i)d_i = \sum_{v \in D_{\leq 2k-1}}(2k - d(v))$$

$$\leq \sum_{v \in D_{\leq 2k-1}} \sum_{e=uv \in E_G(v)} \frac{d(u) - 2k}{d(u)} = \sum_{u \in D_{\geq 2k}} \sum_{\substack{e = uv \in E_G(u) \\ v \in D_{\leq 2k-1}}} \frac{d(u) - 2k}{d(u)}$$

$$\leq \sum_{u \in D_{\geq 2k}} \sum_{e=uv \in E_G(u)} \frac{d(u) - 2k}{d(u)}$$

$$= \sum_{u \in D_{\geq 2k}} (d(u) - 2k) = \sum_{i \geq 2k} (i - 2k)d_i. \tag{8}$$

Combining (7) and (8), it follows that

$$2|E(G)| - 2k(|V(G)| - 1) = 2k + \sum_{i \geq 2k}(i - 2k)d_i - \sum_{i=g}^{2k-1}(2k - i)d_i \geq 2k > 0,$$

contrary to (1). This contradiction establishes Theorem 1.2. □

3. Applications

In this section, we shall show some applications of Theorem 1.2 in the studies of circular flows, spanning connectivity of line graphs and supereulerian width of graphs.

3.1. Applications to circular flow problems

For a graph G, let $D = D(G)$ be an orientation of G. For each vertex v, let $E_D^+(v)$ be the set of arcs in D oriented away from v and $E_D^-(v)$ be the set of arcs in D oriented into v.

Bill Tutte initiated the theory of integer flows as the dual problem and a generalization of the planar map coloring problem. A **nowhere-zero** k-**flow** of a graph G is an orientation D together with a function $f : E(G) \mapsto \{\pm 1, \pm 2, \ldots, \pm(k - 1)\}$ such that $\sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e) = 0$ for each vertex $v \in V(G)$. The concept of circular flow was introduced by Goddyn, Tarsi and Zhang [15] as a generalization of integer flows and a dual version of circular colorings. A **nowhere-zero circular** k/d-**flow** in a graph G is a nowhere-zero k-flow (D, f) such that the range of f is contained in $\{\pm d, \pm(d + 1), \ldots, \pm(k - d)\}$. The **flow index** $\phi(G)$ of a graph G is the least rational number r such that G admits a nowhere-zero circular r-flow. It is proved in [15] that such an index indeed exists, and the circular flow satisfies the monotonicity that for any pair of rational numbers $r \geq s$, a graph admitting a nowhere-zero circular s-flow has a nowhere-zero circular r-flow, as well.

Tutte proposed three outstanding flows conjectures, which are a major source of inspiration in structural graph theory.

- 3-**Flow Conjecture:** For every 4-edge-connected graph G, $\phi(G) \leq 3$.
- 4-**Flow Conjecture:** For every bridgeless graph without Peterson-minor, $\phi(G) \leq 4$.
- 5-**Flow Conjecture:** For every bridgeless graph G, $\phi(G) \leq 5$.

All three conjectures remain open despite massive research efforts and heart-stirring breakthroughs. It was proved by Galluccio and Goddyn [13] that *for every 6-edge-connected graph G, $\phi(G) < 4$*. Lai, Xu and Zhang obtained a half-page concise proof of this result in [26], and in fact, they showed that the conclusion is valid for graphs with 3 edge-disjoint spanning trees.

Thomassen [35] resolved the weak 3-Flow Conjecture by showing *every 8-edge-connected graph admits a nowhere-zero 3-flow*. Lovász, Thomassen, Wu and Zhang [31] further proved *every 6-edge-connected graph admits a nowhere-zero 3-flow*. Those methods are applied in [17] to show *every graph G with $\tau(G) \geq 4$ admits a nowhere-zero 3-flow*. This, together with Theorem 1.2, implies *every 5-edge-connected essentially 23-edge-connected graph admits a nowhere-zero 3-flow*. This verified 3-Flow Conjecture within the family of highly essentially connected 5-edge-connected graphs. Note that a result of Kochol [23] showed the 3-Flow Conjecture is equivalent to its restriction to 5-edge-connected graphs.

The celebrated 5-Flow Conjecture of Tutte is implied by a conjecture of Jaeger [22] that *for every 9-edge-connected graph G, $\phi(G) \leq 5/2$*. Thus studying graphs G with $\phi(G) < 3$ would be of interest. We prove a sufficient condition in terms of spanning tree packing number for a graph G to satisfy $\phi(G) < 3$. Dvořák and Postle [11] obtained a remarkable result on the density of circular 5/2-coloring critical graphs. Utilizing this result and its dual version of circular flow on planar graphs, we further obtain sufficient conditions for a planar graph G to satisfy $\phi(G) \leq 5/2$.

Recall that $\tau(G)$ denotes the maximum number k such that G contains k-edge-disjoint spanning trees. We summarize the above mentioned relationship between $\tau(G)$ and $\phi(G)$ as follows.

Theorem 3.1. *Each of the following holds.*
  (i) [26] For every graph G with $\tau(G) \geq 3$, $\phi(G) < 4$.
  (ii) [17] For every graph G with $\tau(G) \geq 4$, $\phi(G) \leq 3$.
  (iii) For every graph G with $\tau(G) \geq 6$, $\phi(G) < 3$.
  (iv) [11] For every planar graph G with $\tau(G) \geq 5$, $\phi(G) \leq 5/2$.

Applying Theorem 1.2, we further extend these results to certain graphs with highly essential edge connectivity.

Corollary 3.2. *Each of the following holds.*
   (i-a) For every 4-edge-connected essentially 14-edge-connected graph G, $\phi(G) < 4$.
   (i-b) For every 5-edge-connected essentially 11-edge-connected graph G, $\phi(G) < 4$.
   (ii) For every 5-edge-connected essentially 23-edge-connected graph G, $\phi(G) \leq 3$.
   (iii) For every 7-edge-connected essentially 47-edge-connected graph G, $\phi(G) < 3$.
   (iv) For every 8-edge-connected essentially 20-edge-connected planar graph G, $\phi(G) \leq 5/2$.

3.2. Proofs of Theorem 3.1

For disjoint vertex subsets U, W in an undirected graph G, let $[U, W]_G = \{uw \in E(G) | u \in U, w \in W\}$. The subscript G may be omitted when G is understood from the context. Let D be an orientation of G. We use $[U, W]_D$ to denote the set of directed edges in $[U, W]_G$ which are oriented from U to W. Denote $\mathcal{D}(G)$ and $\mathcal{C}(G)$ to be the set of all orientations of G and the set of all edge-cuts of G, respectively. We need the following results on circular flows.

Lemma 3.3 (*Hoffman [19], see [2] p.88, or Theorem 2.3.1 in [43]*)**.** *Let G be a bridgeless graph, D be an orientation of G, and let $a \leq b$ be two positive integers. The following statements are equivalent.*
   (i) For every edge-cut $[A, B] \in \mathcal{C}(G)$,

$$\frac{a}{b} \leq \frac{|[A, B]_D|}{|[B, A]_D|} \leq \frac{b}{a};$$

   (ii) G admits a positive integer flow (D, f) such that $a \leq f(e) \leq b$ for each $e \in E(G)$.

Lemma 3.4 (*Goddyn, Tarsi and Zhang [15]*)**.** *Let G be a bridgeless graph. Then*

$$\phi(G) = \min_{D \in \mathcal{D}(G)} \max_{[A, B] \in \mathcal{C}(G)} \left\{ \frac{|[A, B]_D| + |[B, A]_D|}{|[B, A]_D|} \right\}.$$

Proof of Theorem 3.1(iii). Let $T_1, T_2, \ldots, T_6$ be six edge-disjoint spanning trees of G. For each edge $e \in E(T_2)$, there exists a unique circuit in $T_1 + e$, denoted $C(e)$. Let $C = \triangle_{e \in E(T_2)} C(e)$ be the symmetric difference of all those fundamental circuits. Then C is a connected spanning Eulerian subgraph of G, since $E(T_2) \subset E(C)$. Let (D_2, f_2) be a positive 2-flow of C. Denote $G_1 = G - E(C)$. Since $\cup_{i=3}^{6} E(T_i) \subset E(G_1)$, we have $\tau(G_1) \geq 4$. Hence $\phi(G_1) \leq 3$ by Theorem 3.1(ii). Let (D_1, f_1) be a positive 3-flow of G_1, and let $D = D_1 \cup D_2$. We will show that, under the orientation D of G,

$$\frac{|[A, B]_D| + |[B, A]_D|}{|[B, A]_D|} < 3 \tag{9}$$

for any edge-cut $[A, B] \in \mathcal{C}(G)$.

Firstly, since (D_1, f_1) is a positive 3-flow of G_1, it follows from Lemma 3.3 that

$$\frac{1}{2} \leq \frac{|[A, B]_{D_1}|}{|[B, A]_{D_1}|} \leq 2. \tag{10}$$

For the same reason, we have that in the graph C with orientation D_2,

$$|[A, B]_{D_2}| = |[B, A]_{D_2}| \neq 0. \tag{11}$$

Notice that $|[A, B]_D| = |[A, B]_{D_1}| + |[A, B]_{D_2}|$ and $|[B, A]_D| = |[B, A]_{D_1}| + |[B, A]_{D_2}|$. This, together with (10) and (11), implies that

$$\begin{aligned}
&\frac{|[A, B]_D| + |[B, A]_D|}{|[B, A]_D|} \\
&= 1 + \frac{|[A, B]_{D_1}| + |[A, B]_{D_2}|}{|[B, A]_{D_1}| + |[B, A]_{D_2}|} \\
&= 3 - \frac{|[B, A]_{D_2}|}{|[B, A]_{D_1}| + |[B, A]_{D_2}|} + \frac{|[A, B]_{D_1}| - 2|[B, A]_{D_1}|}{|[B, A]_{D_1}| + |[B, A]_{D_2}|} \\
&< 3.
\end{aligned}$$

Hence (9) holds for any edge-cut $[A, B] \in \mathcal{C}(G)$. Therefore, $\phi(G) < 3$ by Lemma 3.4. $\square$

Remark. Recently, it is proved in [30] that $\phi(G) < 3$ *for every 8-edge-connected graph G* with a more sophisticated method coming from [31,35], and modify the proof in [17], together with the main results in [30], it is possible to show that $\phi(G) < 3$ *for every graph G with $\tau(G) \geq 5$.* (To show this stronger result, the main task is to reduce certain minimal counterexample to a 8-edge-connected graph by using some modified arguments in [17].) But we still think the proof of Theorem 3.1(c) is of value as it only applies a simple and directed argument from a theorem of Hoffman [19].

A **circular** k/d-**coloring** of a graph G is a function $c : V(G) \mapsto \{0, 1, 2, \ldots, k-1\}$ such that for each $uv \in E(G)$, $\min\{|c(u) - c(v)|, |c(u) - c(v) - k|, |c(v) - c(u) - k|\} \geq d$. This coloring function c can be also viewed as a mapping from $V(G)$ to k equally divided points of a circle with circumference k such that the distance between $c(u)$ and $c(v)$ in the circle is at least d for each edge $uv \in E(G)$. Circular coloring was introduced by Vince [38] as a generalization of vertex coloring. The circular flow of graphs was introduced by Goddyn, Tarsi and Zhang in [15] as a dual concept of circular coloring. The following theorem is proved in [15], which generalizes the integer flow and coloring duality theorem of Tutte [36].

Theorem 3.5 (*Goddyn, Tarsi and Zhang [15]*)**.** *Let G be a plane graph and G^* be its dual graph. Then G admits a nowhere-zero circular k/d-flow if and only if G^* admits a circular k/d-coloring.*

A graph G is called $5/2$-*coloring critical* if G has no $5/2$ circular coloring, but every proper subgraph of G has one. The following theorem concerning the density of $5/2$-coloring critical graphs is obtained by Dvořák and Postle [11].

Theorem 3.6 (*Dvořák and Postle [11]*)**.** *For any $5/2$-coloring critical graph G distinct from C_3, we have*

$$5|V(G)| - 4|E(G)| \leq 2.$$

We are now ready to show that Theorem 3.1(iv) follows from Theorem 3.6.

Proof of Theorem 3.1(iv). Suppose, for contradiction, that G admits no nowhere-zero circular $5/2$-flow. Then G^*, the dual of G, has no circular $5/2$-coloring by Theorem 3.5. It follows that G^* contains a $5/2$-coloring critical subgraph H^* (possibly $H^* = G^*$). Notice that the dual graph of H^* corresponds to a contraction of G. More precisely, let $E_0^* = E(G^*) - E(H^*)$, and let E_0 be the set of dual edges of E_0^* in G. Denote $H = G/E_0$. Then H is the dual graph of H^*. Since $\tau(G) \geq 5$, the girth of G^* is at least 5, and so H^* is a triangle-free $5/2$-coloring critical graph. By Theorem 3.6, we have $5|V(H^*)| - 4|E(H^*)| \leq 2$. Since $|E(H)| = |E(H^*)|$ and $|V(H)| + |V(H^*)| - |E(H)| = 2$ by Euler's Formula, we conclude that $|E(H)| \leq 5|V(H)| - 8$. This contradicts the fact that $\tau(H) = \tau(G/E_0) \geq \tau(G) \geq 5$. The proof is completed. $\square$

Remark. Corollary 3.2(iv) also has a dual version as follows: *If G is a plane graph with girth at least 8 and every non-facial cycle in G is of length at least 20, then G admits a circular $5/2$-coloring*. With similar arguments deployed in the proof of Theorem 3.1(c), we also have, by Theorem 3.1(iv), the following corollary: *For every planar graph G with $\tau(G) \geq 7$, $\phi(G) < 5/2$*. Hence we have $\phi(G) < \frac{5}{2}$ for any 14-edge-connected planar graph G.

3.3. Applications to spanning connectivity of line graphs and supereulerian width of graphs

For an integer $s > 0$ and for $u, v \in V(G)$ with $u \neq v$, an $(s; u, v)$-path-system of G is a subgraph H consisting of s internally vertex-disjoint (u, v)-paths, and such an H is called a spanning (s, u, v)-path-system if $V(H) = V(G)$. A graph G is **spanning** s-**connected** if for any $u, v \in V(G)$ with $u \neq v$, G has a spanning $(s; u, v)$-path-system. The **spanning connectivity** $\kappa^*(G)$ of a graph G is the largest integer s such that for any integer k with $1 \leq k \leq s$ and for any $u, v \in V(G)$ with $u \neq v$, G has a spanning $(k; u, v)$-path-system. A graph G is Hamiltonian-connected if for any $u, v \in V(G)$ with $u \neq v$, G has a path P from u to v such that $V(P) = V(G)$. If G is not Hamiltonian-connected, then $\kappa^*(G) = 0$. Thus $\kappa^*(G) \geq 1$ if and only if G is Hamiltonian-connected. The Hamiltonian-connectedness of graphs has been intensively studied, as shown in [16]. The spanning connectivity of a graph has also been widely studied, as can be seen in Chapters 14 and 15 of [20].

The **line graph** of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G share a vertex in $L(G)$. There have been studies on the spanning connectivity of line graphs. The next theorem is a summary.

Theorem 3.7. *Let G be a graph. Each of the following holds.*
(i) (Zhan [41]) *If $\kappa'(G) \geq 4$, then $\kappa^*(L(G)) \geq 2$.*
(ii) (Catlin and Lai [6]) *If $\tau(G) \geq 2$, then $\kappa^*(L(G)) \geq 2$ if and only if $\kappa(L(G)) \geq 3$.*
(iii) (Zhan [42]) *If $\kappa'(G) \geq 3$ and G is essentially 7-edge-connected, then $\kappa^*(L(G)) \geq 2$.*
(iv) (Huang and Hsu [21], Chen et al., Theorem 1.4 of [9]) *For any integer $k \geq 2$, if $\tau(G) \geq k$, then $\kappa^*(L(G)) \geq k$.*

Applying Theorem 1.2 with $k = 2$ and $g = 3$, it follows that every 3-edge-connected, essentially 7-edge-connected graph has 2 edge-disjoint spanning trees. Thus Theorem 3.7(iii) follows from Theorems 1.2 and 3.7(ii). It is known that Theorem 3.7 (iii) implies that every 7-connected line graph is Hamiltonian-connected. The following is an immediate corollary of Theorems 1.2 and 3.7(iv), which is equivalent to the statement (v) in abstract as an essential edge-cut of H corresponds to a vertex cut in $L(H)$.

Corollary 3.8. *Let g and k be integers with $2k - 1 \geq g \geq k + 1 \geq 3$. The line graph of a g-edge-connected, essentially $(\lceil \frac{g^2}{g-k} \rceil - 2)$-edge-connected graph is spanning k-connected.*

Let G be a graph, and $s > 0$ be an integer. For any distinct $u, v \in V(G)$, an $(s; u, v)$-**trail-system** of G is a subgraph H consisting of s edge-disjoint (u, v)-trails. A graph is **supereulerian with width** s if for any $u, v \in V(G)$ with $u \neq v$, G has a spanning $(s; u, v)$-trail-system. The **supereulerian width** $\mu'(G)$ of a graph G is the largest integer s such that G is supereulerian with width k for any integer k with $1 \leq k \leq s$. As indicated in [9,27,29], there have been many studies on supereulerian problem, which is the cases for graphs G with $\mu'(G) \geq 2$. The supereulerian width problem is a natural generalization of the supereulerian problem, which seeks the existence of a spanning Eulerian subgraph, is proposed by Boesch, Suffel, and Tindell in [3]. For some recent progress on the supereulerian problem, see Catlin's survey [5] and its updates [10,25]. A reduction method to study $\mu'(G)$ has been developed in [29] and [27]. The following is obtained.

Theorem 3.9 (*Li et al., Theorem 2.11 and Corollary 2.9(iii) of [29]*)**.** *Let G be a graph. If $\tau(G) \geq 2$, then $\mu'(G) \geq \tau(G)$.*

Applying Theorems 1.2 and 3.9, we have the following corollary.

Corollary 3.10. *For any integers g and k with $k \geq 2$ and $g \geq k + 1$, if G is g-edge-connected and essentially $(\lceil \frac{g^2}{g-k} \rceil - 2)$-edge-connected, then $\mu'(G) \geq k$.*

4. Concluding remarks

Flow with boundaries is also related to claw decomposition and generalized claw decomposition, as indicated by Barát and Thomassen [1]. Several similar results on claw decomposition can be modified for highly essentially connected graphs via Theorem 1.2. We omit those statements and leave it for interested readers.

Let $t > 2p + 1$ be an integer. Denote by $K_{2p+1,t}^+$ the graph obtained from the complete bipartite graph $K_{2p+1,t}$ by adding a new edge connecting two degree t vertices. It is straightforward to verify that $K_{2p+1,t}^+$ does not admit a modulo $(2p + 1)$-orientation, and so $\phi(K_{2p+1,t}^+) > 2 + 1/p$. Notice that $K_{2p+1,t}^+$ is $(2p + 1)$-edge-connected essentially $(t + 2p - 1)$-edge-connected. Therefore, there exist $(2p + 1)$-edge-connected graphs with arbitrary highly essential edge connectivity such that the flow index is greater than $2 + 1/p$. This leads to the following question.

Problem 4.1. Does there exist a constant $h = h(p)$ such that $\phi(G) \leq 2 + 1/p$ for every $(2p + 2)$-edge-connected essentially h-edge-connected graph G?

Tutte's 3-Flow Conjecture, if true, would imply a positive answer for the case of $p = 1$. Not much is known for $p \geq 2$. We believe the following weaker versions are true.

Conjecture 4.2 (a)**.** *For every integer $p \geq 1$, there exists a constant $h = h(p)$ such that $\phi(G) \leq 2 + 1/p$ for every $4p$-edge-connected essentially h-edge-connected graph G.*

(b) For every integer $p \geq 1$, there exists a constant $h = h(p)$ such that $\phi(G) < 2 + 1/p$ for every $(4p + 2)$-edge-connected essentially h-edge-connected graph G.

The results in [17] provide some supporting evidence for the $p = 1$ case of Conjecture 4.2(a), and the $p = 1$ case of Conjecture 4.2(b) is known to be true by Theorem 1.2 and a previous remark saying that *every graph G with $\tau(G) \geq 5$ has flow index $\phi(G) < 3$*. Conjecture 4.2 is open so far for every $p \geq 2$. Following the method in [17] with more effort, together with some results in [31], it should be possible to show $\phi(G) \leq 5/2$ for every graph G with $\tau(G) \geq 8$, and the same should be true for 9-edge-connected graph with high essential edge connectivity by Theorem 1.2. This also provides some partial evidence for the $p = 2$ case of Conjecture 4.2.

The circular flow conjecture of Jaeger [22] states that $\phi(G) \leq 2 + 1/p$ for every $4p$-edge-connected graph G. This conjecture was disproved in [18] very recently. Infinitely many $4p$-edge-connected counterexamples (for $p \geq 3$) and $(4p + 1)$-edge-connected counterexamples (for $p \geq 5$) are constructed in [18]. However, all those constructed counterexamples contain many essential edge-cuts of small size. For this reason, we still believe the truth of Conjecture 4.2.

Acknowledgments

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