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Counterexamples to Jaeger's Circular Flow Conjecture

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It was conjectured by Jaeger that *every* $4p$*-edge-connected graph admits a modulo* $(2p+1)$*-orientation (and, therefore, admits a nowhere-zero circular* $(2+\frac{1}{p})$*-flow).* This conjecture was partially proved by Lovász et al. (2013) [7] for $6p$-edge-connected graphs. In this paper, infinite families of counterexamples to Jaeger's conjecture are presented. For $p \geq 3$, there are $4p$-edge-connected graphs not admitting modulo $(2p+1)$-orientation; for $p \geq 5$, there are $(4p+1)$-edge-connected graphs not admitting modulo $(2p+1)$-orientation.

1. Introduction

In 1981, Jaeger [3] (see also [4]) proposed the following conjecture, known as Circular Flow Conjecture, or Modulo Orientation Conjecture.

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Conjecture 1.1 *(Jaeger's Circular Flow Conjecture). Every $4p$-edge-connected graph admits a modulo $(2p+1)$-orientation.*

In [5], Kochol also suggested a seemly weaker conjecture.

Conjecture 1.2. *Every $(4p+1)$-edge-connected graph admits a modulo $(2p+1)$-orientation.*

For $p = 1$, Kochol [5] showed that both Conjecture 1.1 and Conjecture 1.2 are equivalent to the 3-Flow Conjecture of Tutte. In the case of $p = 2$, the truth of Conjecture 1.2 (and Conjecture 1.1) would imply Tutte's 5-Flow Conjecture (see [4,5]).

Resolving the weak 3-flow conjecture and the weak circular flow conjecture, Thomassen [9] showed that such orientation exists under the edge connectivity 8 ($p = 1$) and $2(2p+1)^2 + 2p + 1$ ($p \geq 2$), respectively. Lovász et al. [7] further proved that every $6p$-edge-connected graph admits a modulo $(2p+1)$-orientation.

In this paper, we construct a $4p$-edge-connected graph without modulo $(2p+1)$-orientation for every $p \geq 3$. Furthermore, for every $p \geq 5$, we also construct a $(4p+1)$-edge-connected graph without modulo $(2p+1)$-orientation. This disproves Jaeger's Circular Flow Conjecture (Conjecture 1.1) for every $p \geq 3$ and Conjecture 1.2 for every $p \geq 5$.

Theorem 1.3. *For every integer $p \geq 3$, there exists a $4p$-edge-connected graph admitting no modulo $(2p+1)$-orientation.*

Theorem 1.4. *For every integer $p \geq 5$, there exists a $(4p+1)$-edge-connected graph admitting no modulo $(2p+1)$-orientation.*

In Section 5, graphs constructed in Theorems 1.3 and 1.4 are further extended to infinite families of counterexamples to Conjectures 1.1 and 1.2.

We shall present the construction of Theorem 1.3 first, which is simpler to analyze. The construction in Theorem 1.4 is based on the same idea with some more elaborate modification.

2. Preliminary

Graphs in this paper are finite and may contain parallel edges. In an undirected graph G, for vertex subsets $U, W \subseteq V(G)$, let $[U, W]_G = \{uw \in E(G) : u \in U, w \in W\}$ and $\delta_G(U) = [U, V(G) - U]_G$. For $v, w \in V(G)$, define $E_G(v) = [\{v\}, V(G) - \{v\}]_G$ and $E_G(v, w) = [\{v\}, \{w\}]_G$, respectively. An edge-cut X of G is called *trivial* if $X = E_G(v)$ for some $v \in V(G)$, and *nontrivial* otherwise. Let $D = D(G)$ be an orientation of G. If $A \subset V(G)$, we define $E^+_D(A)$ ($E^-_D(A)$, respectively) to be the set of all directed edges with initial vertex (terminal vertex, respectively) in A and terminal vertex (initial vertex, respectively) in $V(G) - A$. When $A = \{v\}$, We simply use $E^+_D(v)$ and $E^-_D(v)$ for convenience. For vertex subsets $U, W \subseteq V(G)$, we denote $[U, W]_D = E^+_D(U) \cap E^-_D(W)$.

In addition, $d_G(v) = |E_G(v)|$, $d^-_D(v) = |E^-_D(v)|$ and $d^+_D(v) = |E^+_D(v)|$ are known as the degree, indegree and outdegree of a vertrex v, respectively.

A graph G admits a *modulo* $(2p+1)$-*orientation* if it has an orientation D such that $d^+_D(v) - d^-_D(v) \equiv 0 \pmod{2p+1}$ for each $v \in V(G)$. It is observed by Jaeger [4] that a graph admits a nowhere-zero circular $(2 + \frac{1}{p})$-flow if and only if it admits a modulo $(2p+1)$-orientation. In particular, a graph has a nowhere-zero 3-flow if and only if it admits a modulo 3-orientation. The readers are referred to [10] for a comprehensive introduction on nowhere-zero flows.

Observation 2.1. *Let $F = (2p-1)K_2$ be the graph consisting of two vertices u, v and $2p-1$ parallel edges between u and v, and, let $t \in Z_{2p+1}$. The graph F admits an orientation D such that*

$$d^+_D(u) - d^-_D(u) \equiv t \pmod{2p+1}$$

if and only if $t \neq 0$.

Proof. It is obvious that there is no such orientation for $t = 0$. The existence of such an orientation is essentially a solution of the following equations

$$\begin{cases} d^+_D(u) - d^-_D(u) \equiv t \pmod{2p+1}, \\ d^+_D(u) + d^-_D(u) = 2p-1. \end{cases}$$

For $t \in \{1, \cdots, 2p\}$, an orientation D of F such that

$$d^+_D(u) = |E^+_D(u)| = \begin{cases} p + \frac{t-1}{2} & \text{if } t \text{ is odd}, \\ \frac{t}{2} - 1 & \text{if } t \text{ is even}, \end{cases}$$

and $d^-_D(u) = |E^-_D(u)| = (2p-1) - |E^+_D(u)|$ would be sufficient. □

Our construction relies on the following 2-sum operation, which generalizes the "edge superposition" method in [6]. In fact, the case $p = 1$ of Lemma 2.3 below coincides with Proposition 4.6 in [6] or Lemma 1 in [5].

Definition 2.2. Let H_1 and H_2 be two graphs with $u_1, v_1 \in V(H_1)$, $u_2, v_2 \in V(H_2)$ and $|E_{H_1}(u_1, v_1)| \geq 2p-1$. Define $H = H_1 \oplus_2 H_2$, the **2-sum of** H_1 **and** H_2, to be the graph obtained from H_1 and H_2 by deleting $2p-1$ parallel edges between u_1 and v_1 in H_1, and then identifying u_1 and u_2 to be a new vertex u, and identifying v_1 and v_2 to be a new vertex v (see Fig. 1).

Lemma 2.3. *Let $H = H_1 \oplus_2 H_2$ be a 2-sum of H_1 and H_2 used in Definition 2.2. If neither H_1 nor H_2 admits a modulo $(2p+1)$-orientation, then $H = H_1 \oplus_2 H_2$ admits no modulo $(2p+1)$-orientation.*

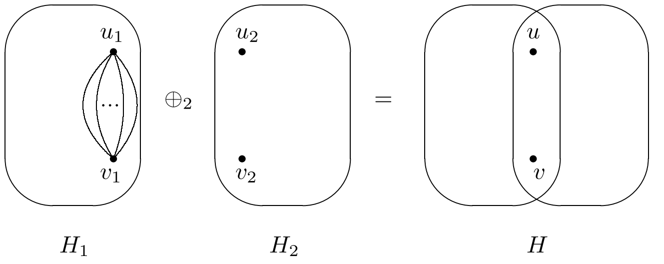

Fig. 1. The 2-sum of H_1 and H_2.

Proof. Let $u, v \in V(H)$, $u_i, v_i \in V(H_i)$ $(i = 1, 2)$ be the vertices described in Definition 2.2, and let F be the set of $(2p-1)$ parallel edges of H_1 deleted in the 2-sum.

Suppose that H admits a modulo $(2p+1)$-orientation D. Let D_2 be the restriction of D on H_2 and D_1 be the restriction of D on $H_1 - F$. Let $\beta_i(u_i) = d^+_{D_i}(u_i) - d^-_{D_i}(u_i)$ and $\beta_i(v_i) = d^+_{D_i}(v_i) - d^-_{D_i}(v_i)$, for each $i = 1, 2$. It is obvious that

$$\beta_1(u_1) \equiv -\beta_1(v_1) \equiv -\beta_2(u_2) \equiv \beta_2(v_2) \pmod{2p+1}.$$

Since H_2 does not admit a modulo $(2p+1)$-orientation, $\beta_2(u_2) \equiv -\beta_2(v_2) \not\equiv 0 \pmod{2p+1}$. By Observation 2.1, the edge subset F can be properly oriented so that the resulting orientation (together with D_1) is a modulo $(2p+1)$-orientation of H_1. This is a contradiction. $\square$

3. The constructions of counterexamples – proof of Theorem 1.3

3.1. Step 1 of the construction

It is known that the complete graph K_{4p} admits no modulo $(2p+1)$-orientation. Our first construction starts from it.

Construction 1. *Let $p \geq 3$ be an integer, and $\{v_1, \cdots, v_{4p}\}$ be the vertex set of the complete graph K_{4p}.*

(i) Construct a graph G_1 from the complete graph K_{4p} by adding an additional set T of edges such that $V(T) = \{v_1, \cdots, v_{3(p-1)}\}$ and each component of the edge-induced subgraph $G_1[T]$ is a triangle (see G_1 in Fig. 2).

(ii) Construct a graph G_2 from G_1 by adding two new vertices z_1 and z_2, adding one edge z_1z_2, adding $(p-2)$ parallel edges connecting v_{4p} and z_j for $j = 1, 2$, and adding one edge v_iz_j for each $3p-2 \leq i \leq 4p-1$ and $j = 1, 2$ (see G_2 in Fig. 2).

Lemma 3.1. *(i) G_1 admits no modulo $(2p+1)$-orientation.*
(ii) G_2 admits no modulo $(2p+1)$-orientation. Moreover, G_2 contains exactly two edge-cuts, $E(z_1), E(z_2)$, of sizes $2p+1$, and all the other edge-cuts are of sizes at least $4p$.

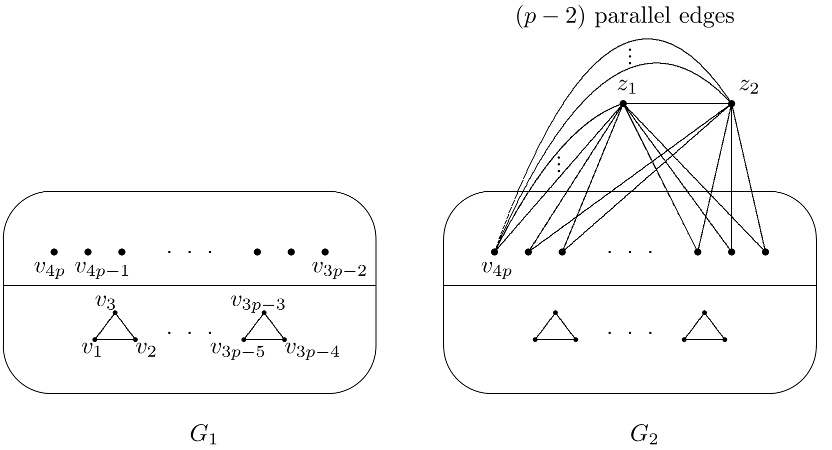

Fig. 2. The graphs G_1 and G_2.

Proof. (i) Suppose to the contrary that G_1 admits a modulo $(2p+1)$-orientation D. Notice that $d_D^+(v) - d_D^-(v) \in \{\pm(2p+1)\}$ for each vertex $v \in V(G_1)$. Denote $V^+ = \{x \in V(G_1) : d_D^+(x) - d_D^-(x) = 2p+1\}$ and $V^- = \{x \in V(G_1) : d_D^+(x) - d_D^-(x) = -2p-1\}$, respectively. Clearly, $|V^+| = |V^-| = 2p$. Since the edge-induced subgraph $G_1[T]$ consists of $(p-1)$ vertex-disjoint triangles, each of which may contribute at most two edges in the edge-cut $[V^+, V^-]_{G_1}$, we have

$$|[V^+, V^-]_{G_1}| \le |V^+| \cdot |V^-| + 2(p-1) = 4p^2 + 2p - 2 < 4p^2 + 2p.$$

This contradicts to the fact that

$$\begin{aligned} 4p^2 + 2p = |V^+| \cdot (2p+1) = \sum_{v \in V^+} (d_D^+(v) - d_D^-(v)) = |[V^+, V^-]_D| - |[V^-, V^+]_D| \\ \le |[V^+, V^-]_{G_1}|. \end{aligned}$$

(ii) The proof is by contradiction. Suppose that G_2 admits a modulo $(2p+1)$-orientation D. Without loss of generality, assume the edge $z_1 z_2$ is oriented from z_1 to z_2 under the orientation D. Thus, $|E_D^+(z_1)| = |E_{G_2}(z_1)| = 2p+1$ and $|E_D^-(z_2)| = |E_{G_2}(z_2)| = 2p+1$. Furthermore, since $|E_{G_2}(z_1, v_i)| = |E_{G_2}(z_2, v_i)|$ for each $3p-2 \le i \le 4p$, the restriction of D on $E(G_2) - E(G_1)$ is a modulo $(2p+1)$-orientation, and, therefore, the restriction of D on $E(G_1)$ is also a modulo $(2p+1)$-orientation. This contradicts (i). □

3.2. Step 2 of the construction

Construction 2. *Denote by C_{4p+1} the cycle of length $4p+1$ with $V(C_{4p+1}) = \{c_i : i \in Z_{4p+1}\}$ and $E(C_{4p+1}) = \{c_i c_{i+1} : i \in Z_{4p+1}\}$. Let $W = (2p-1)C_{4p+1} \cdot K_1$ be the graph*

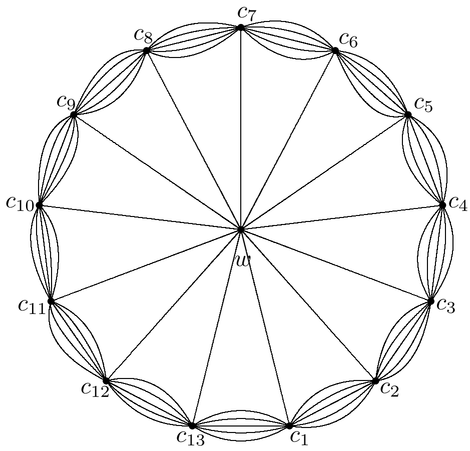

Fig. 3. The graph W for $p = 3$.

obtained from C_{4p+1} *by replacing each edge* c_ic_{i+1} *with* $2p-1$ *parallel edges, and then adding a center vertex* w *joining each vertex* c_i *in the cycle (see Fig. 3).*

We remark that the graph W is the dual of an example discovered by DeVos in [2] (also see [1]) on the circular coloring of planar graphs. We include a proof of the following lemma for the purpose of self-completeness.

Lemma 3.2. *The graph* W *admits no modulo* $(2p+1)$*-orientation. Moreover,* W *is* $(4p-1)$*-edge-connected and every* $(4p-1)$*-edge-cut is trivial.*

Proof. Suppose that W admits a modulo $(2p+1)$-orientation D. Notice that, for each vertex c_i, $d^+_D(c_i) - d^-_D(c_i) = 2p+1$ or $= -(2p+1)$. Furthermore, since the cycle C_{4p+1} is of odd length, there exists two consecutive vertices c_i, c_{i+1} in the cycle with $d^+_D(c_i) - d^-_D(c_i) = d^+_D(c_{i+1}) - d^-_D(c_{i+1})$ $(\in \{\pm(2p+1)\})$. However,

$$\begin{aligned} 4p+2 &= |(d^+_D(c_i) - d^-_D(c_i)) + (d^+_D(c_{i+1}) - d^-_D(c_{i+1}))| \\ &= ||E^+_D(\{c_i, c_{i+1}\})| - |E^-_D(\{c_i, c_{i+1}\})|| \\ &\leq |\delta_W(\{c_i, c_{i+1}\})| = 4p < 4p+2, \end{aligned}$$

a contradiction. □

3.3. The final step of the construction

Now, we are ready to obtain our final construction via the 2-sum operations of W and copies of G_2.

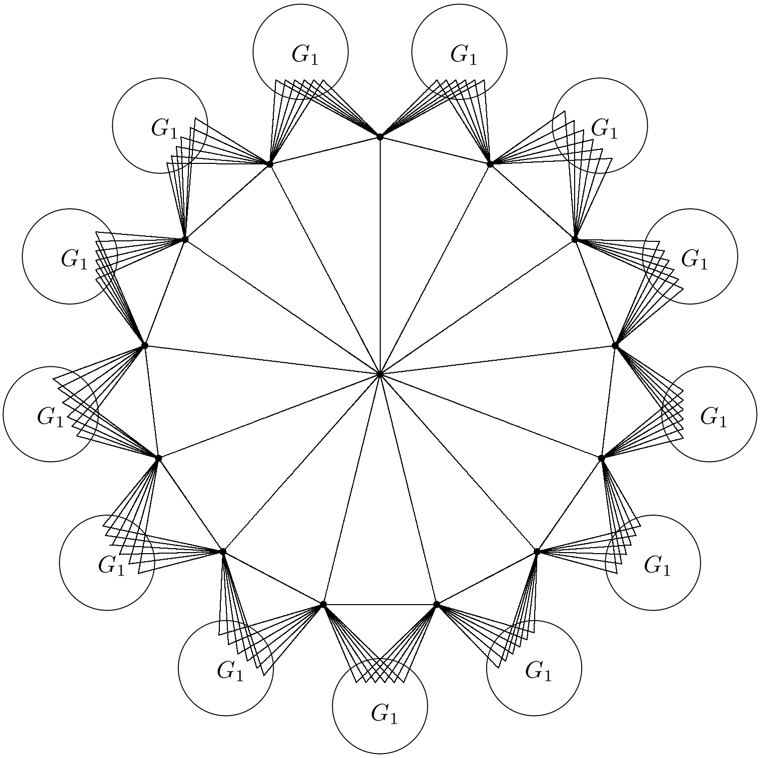

Fig. 4. The graph M for $p = 3$.

Construction 3. *For each c_i, c_{i+1} ($i \in Z_{4p+1}$) in W and z_1, z_2 in a copy of G_2, apply the 2-sum operation described in Definition 2.2. Denote M to be the final graph obtained after these $4p+1$ 2-sum operations (see Fig. 4).*

Lemma 3.3. *The graph M is $4p$-edge-connected and admits no modulo $(2p+1)$-orientation.*

Proof. It is straightforward to check M is $4p$-edge-connected. Specifically, every vertex in M is of degree at least $4p+1$. If a nontrivial edge-cut Q separates z_1 and z_2 in a copy of G_2, then Q must separate at least two copies of G_2 since it intersects the cycle C_{4p+1} even number of times. In each copy, at least $2p+1$ edges is contained in the cut Q, resulting that Q is of size at least $4p+2$. If a nontrivial edge-cut Q does not separate z_1 and z_2 in any copy of G_2, then Q contains an edge-cut $Q' \neq E_{G_2}(z_1), E_{G_2}(z_2)$ in a copy of G_2, which is of size at least $4p$. Therefore, M is $4p$-edge-connected.

By Lemmas 3.1 and 3.2 and applying Lemma 2.3 consecutively, M admits no modulo $(2p+1)$-orientation. This completes the proof of Lemma 3.3, as well as Theorem 1.3. □

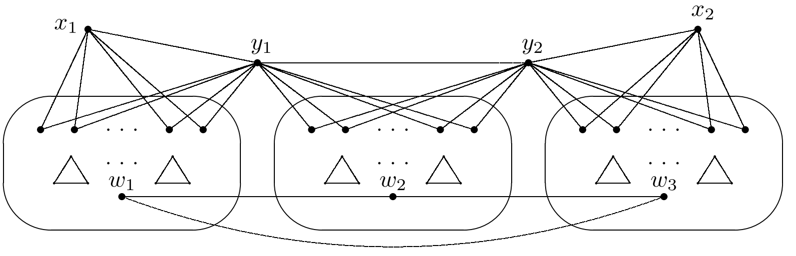

Fig. 5. The graph G_3.

4. The constructions of counterexamples – proof of Theorem 1.4

Note that each $4p$-edge-cut in M is of the form $\delta_M(G_1)$ for some copy of G_1. In this section, the Construction 1 is refined for constructing a new graph G_3, which eliminates these $4p$-edge-cuts. However, the lower bound of p is unavoidably raised to 5 in the new construction.

Construction 4. *Let $p \geq 5$ be an integer, and $\{v_1, \cdots, v_{4p}\}$ be the vertex set of the complete graph K_{4p}. Let $q = \lceil \frac{2p-1}{3} \rceil$.*

(i) Construct a graph G_1' from the complete graph K_{4p} by adding an additional set T' of edges such that $V(T') = \{v_1, \cdots, v_{3q}\}$ and each component of the edge-induced subgraph $G_1'[T']$ is a triangle.

(ii) Construct a graph G_2' from G_1' by adding two new vertices z_1' and z_2', adding one edge $z_1'z_2'$, adding $(3q - 2p + 2)$ parallel edges connecting v_{4p-1} and z_j' for $j = 1, 2$, and adding one edge $v_i z_j'$ for each $3q + 1 \leq i \leq 4p - 2$ and $j = 1, 2$.

(iii) Let G_2^1, G_2^2, G_2^3 be three copies of G_2'. Construct a graph G_3 from these three copies of G_2' by identifying the corresponding z_1' in G_2^1 and G_2^2 to be a new vertex y_1, identifying the corresponding z_2' in G_2^2 and G_2^3 to be a new vertex y_2, and adding a triangle connecting the corresponding v_{4p}'s of G_2^1, G_2^2 and G_2^3. Relabel the corresponding v_{4p}'s of G_2^1, G_2^2 and G_2^3 as w_1, w_2, w_3, and relabel the remaining two degree $2p + 1$ vertices as x_1, x_2, respectively (see Fig. 5).

Lemma 4.1. *(i) Neither G_1' nor G_2' admit a modulo $(2p + 1)$-orientation.*
(ii) G_3 admits no modulo $(2p+1)$-orientation. In addition, G_3 is $(2p+1)$-edge-connected, and each edge-cut that does not separate $\{x_1, x_2\}$ is of size at least $4p + 1$.

Proof. (i) The proof of (i) is analogous to that of Lemma 3.1 (i). Suppose that D is a modulo $(2p + 1)$-orientation of G_1'. With a similar setting as in Lemma 3.1, we have

$$|[V^+, V^-]_{G_1'}| \leq |V^+| \cdot |V^-| + 2\lceil \frac{2p-1}{3} \rceil < 4p^2 + 2p.$$

This contradicts to the fact that

$$4p^2 + 2p = |V^+| \cdot (2p+1) = \sum_{v \in V^+} (d_D^+(v) - d_D^-(v)) = |[V^+, V^-]_D| - |[V^-, V^+]_D|$$
$$\leq |[V^+, V^-]_{G_1'}|.$$

The argument for G_2' is the same as G_2. Note that $d_{G_2'}(z_1') = d_{G_2'}(z_2') = 2p+1$.

(ii) The proof is by contradiction. Suppose that G_3 admits a modulo $(2p+1)$-orientation D. Let D_i be the restriction of D on G_2^i, for $i = 1, 2, 3$.

We first claim that, *under the orientation D, the edges w_1w_2, w_1w_3 are either both oriented away from w_1 or both oriented towards w_1*. If not, since $\{w_1w_2, w_1w_3\}$ is oriented with opposite directions at w_1, we have, under the orientation D_1 of G_2^1,

$$d_{D_1}^+(w_1) - d_{D_1}^-(w_1) \equiv 0 \pmod{2p+1}.$$

Then it follows that

$$d_{D_1}^+(y_1) - d_{D_1}^-(y_1) \equiv -\sum_{v \in V(G_2^1)\setminus\{y_1\}} (d_{D_1}^+(v) - d_{D_1}^-(v)) \equiv 0 \pmod{2p+1}.$$

This implies D_1 is a modulo $(2p+1)$-orientation of G_2^1, yielding a contradiction to (i). Similar conclusion holds for w_3.

Without loss of generality, we assume the edges w_1w_2, w_1w_3 are both oriented away from w_1 in the orientation D. Symmetrically, both edges w_1w_3 and w_2w_3 are oriented towards w_3 in D.

Since $E_{G_2^2}(y_1) \cup \{w_1w_2, w_1w_3\}$ is an edge-cut of G_3, it follows from the orientations of w_1w_2 and w_1w_3 that

$$d_{D_2}^+(y_1) - d_{D_2}^-(y_1) + 2 \equiv 0 \pmod{2p+1},$$

and symmetrically,

$$d_{D_2}^+(y_2) - d_{D_2}^-(y_2) - 2 \equiv 0 \pmod{2p+1}.$$

Since $d_{G_2^2}(y_1) = d_{G_2^2}(y_2) = 2p+1$ in G_2^2, we have

$$d_{D_2}^+(y_1) - d_{D_2}^-(y_1) = -(d_{D_2}^+(y_2) - d_{D_2}^-(y_2)) = 2p - 1. \tag{1}$$

Let $V^+ = \{x \in V(G_2^2) : d_{D_2}^+(x) - d_{D_2}^-(x) > 0\}$ and $V^- = \{x \in V(G_2^2) : d_{D_2}^+(x) - d_{D_2}^-(x) < 0\}$. Then $\{V^+, V^-\}$ is a partition of $V(G_2^2)$ as each vertex of G_2^2 is of odd degree. Clearly, $d_{D_2}^+(w_2) - d_{D_2}^-(w_2) \in \{\pm(2p+1)\}$ by the orientations of w_1w_2 and w_2w_3.

Since $d_{D_2}^+(v_{4p-1}) - d_{D_2}^-(v_{4p-1}) \equiv 0 \pmod{2p+1}$ and

$$d_{G_2^2}(v_{4p-1}) = 4p - 1 + 2(3q - 2p + 2) = 6\lceil \frac{2p-1}{3} \rceil + 3 < 3(2p+1),$$

we have $d_{D_2}^+(v_{4p-1}) - d_{D_2}^-(v_{4p-1}) \in \{\pm(2p+1)\}$ as well.

So, we conclude that

$$d^+_{D_2}(x) - d^-_{D_2}(x) = 2p + 1, \ for\ each\ vertex\ x \in V^+ \setminus \{y_1\}, \tag{2}$$

$$d^+_{D_2}(x) - d^-_{D_2}(x) = -2p - 1, \ for\ each\ vertex\ x \in V^- \setminus \{y_2\}, \tag{3}$$

and

$$|V^+| = |V^-| = 2p + 1. \tag{4}$$

Let S be the set of edge-disjoint 2-paths of G_2^2 joining y_1 and y_2, where $|S| = 2p$. Note that each 2-path in S contributes one edge in the edge-cut $[V^+, V^-]_{G_2^2}$, and $G_2^2[T']$ consists of q triangles, each of which may contribute at most two edges in the edge-cut $[V^+, V^-]_{G_2^2}$. Thus, we have

$$\begin{aligned} |[V^+, V^-]_{G_2^2}| &\le (|V^+| - 1)(|V^-| - 1) + 2q + |S| + |E(y_1, y_2)| \\ &= (2p)^2 + 2\lceil \frac{2p-1}{3} \rceil + 2p + 1 \\ &< 4p^2 + 4p - 1. \qquad\qquad \text{(by } p \ge 5\text{)} \end{aligned}$$

However, by Eq. (1), (2), (3) and (4), we obtain a contradiction as follows.

$$4p^2 + 4p - 1 = (2p + 1)|V^+ \setminus \{y_1\}| + 2p - 1 = \sum_{x \in V^+} (d^+_{D_2}(x) - d^-_{D_2}(x)) \le |[V^+, V^-]_{G_2^2}|.$$

This proves (ii). □

The next construction is similar to Construction 3, except that we replace copies of G_2 with copies of G_3.

Construction 5. *Construct a graph M' as follows: Take $4p + 1$ copies of G_3, then for each c_i, c_{i+1} ($i \in Z_{4p+1}$) in W and x_1, x_2 in a copy of G_3, apply the 2-sum operation described in Definition 2.2.*

The following lemma is a mimic of Lemma 3.3, which eliminates $4p$-edge-cuts.

Lemma 4.2. *For every $p \ge 5$, the graph M' is $(4p + 1)$-edge-connected and admits no modulo $(2p + 1)$-orientation.*

Proof. M' admits no modulo $(2p+1)$-orientation for the same reason as in Lemma 3.3. Similar argument applies to check that M' is $(4p + 1)$-edge-connected. Notice that, by Lemma 4.1, each edge-cut in G_3 that does not separate $\{x_1, x_2\}$ is of size at least $4p+1$. This proves Lemma 4.2, as well as Theorem 1.4. □

5. Remarks

The counterexamples constructed in Theorems 1.3 and 1.4 can be easily extended to some infinite families of counterexamples. One of the most straightforward methods is to replace some vertices of the graphs M and M' by copies of some highly connected graphs (such as, complete graphs of large orders), and see [6] for a similar "vertex superposition" method. Another method is to replace the cycle C_{4p+1} in Construction 2 with a longer odd cycle. We may also apply the 2-sum operations on copies of W, and then modify the final construction. In addition, for the final construction, it is not necessary to apply the 2-sum operation for each c_i, c_{i+1} ($i \in Z_{4p+1}$) in W, as long as there is no vertex of degree $4p-1$ in the resulting graph, it produces a $4p$-edge-connected graph (or $(4p+1)$-edge-connected graph in Construction 5, respectively). Applying the splitting theorem of Mader [8] would yield a $4p$-edge-connected (or $(4p+1)$-edge-connected, for $p \geq 5$, respectively) $(4p+1)$-regular graph without modulo $(2p+1)$-orientation as well. We leave all those details to interested readers.

The construction in this paper seems to suggest that the gap between $4p$ and edge connectivity for admitting modulo $(2p+1)$-orientation may depend on p. Therefore, we propose the following new conjecture on modulo orientations, whose truth still implies the 3-Flow Conjecture and 5-Flow Conjecture of Tutte, as shown by Kochol [5] and Jaeger [4].

Conjecture 5.1. *For every positive integer p, there exists a sufficiently small positive constant $\varepsilon = \varepsilon(p) < \frac{1}{2}$ such that every $\lceil (4+\varepsilon)p \rceil$-edge-connected graph admits a modulo $(2p+1)$-orientation.*

Theorem 1.4 indicates $\varepsilon(p) > \frac{1}{p}$ when $p \geq 5$.

References

[1] O.V. Borodin, S.-J. Kim, A.V. Kostochka, D.B. West, Homomorphisms from sparse graphs with large girth, J. Combin. Theory Ser. B 90 (2004) 147–159.
[2] M. DeVos, Communication at Workshop on Flows and Cycles, Simon Fraser University, June 2000.
[3] F. Jaeger, On circular flows in graphs, in: Finite and Infinite Sets, Vol. I, II, Eger, 1981, in: Colloq. Math. Soc. János Bolyai, vol. 37, North-Holland, Amsterdam, 1984, pp. 391–402.
[4] F. Jaeger, Nowhere-zero flow problems, in: L.W. Beineke, R.J. Wilson (Eds.), Selected Topics in Graph Theory 3, Academic Press, London, 1988, pp. 71–95.
[5] M. Kochol, An equivalent version of the 3-flow conjecture, J. Combin. Theory Ser. B 83 (2001) 258–261.
[6] M. Kochol, Superposition and constructions of graphs without nowhere-zero k-flows, European J. Combin. 23 (2002) 281–306.
[7] L.M. Lovász, C. Thomassen, Y. Wu, C.-Q. Zhang, Nowhere-zero 3-flows and modulo k-orientations, J. Combin. Theory Ser. B 103 (2013) 587–598.
[8] W. Mader, A reduction method for edge-connectivity in graphs, Ann. Discrete Math. 3 (1978) 145–164.
[9] C. Thomassen, The weak 3-flow conjecture and the weak circular flow conjecture, J. Combin. Theory Ser. B 102 (2012) 521–529.
[10] C.-Q. Zhang, Integer Flows and Cycle Covers of Graphs, Marcel Dekker, New York, 1997.