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The flow index and strongly connected orientations

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ABSTRACT

We prove that, for any natural number p, the flow index $\phi(G) < 2 + \frac{1}{p}$ if and only if G has a strongly connected modulo $(2p+1)$-orientation. For the case $p = 1$ we prove that the flow index of every 8-edge-connected graph is strictly less than 3.

1. Introduction

Jaeger's *circular flow conjecture* [6] says that, for every natural number p, every $4p$-edge-connected graph admits a modulo $(2p+1)$-orientation, that is, an orientation where each vertex has the same indegree as outdegree modulo $(2p+1)$. For $p = 1$ this implies Tutte's 3-*flow conjecture*, and for $p = 2$ it implies Tutte's 5-*flow conjecture*. It was observed by Jaeger [6] that a graph G admits a modulo $(2p+1)$-orientation if and only if the graph G admits a circular $(2+\frac{1}{p})$-flow, to be defined below. This suggests the definition of the *flow index* $\phi(G)$ of a bridgeless graph G, as the smallest rational number q such that G admits a circular q-flow. Thus Tutte's flow conjectures says that, for every 4-edge-connected graph G, $\phi(G) \leq 3$, and, for every 2-edge-connected graph G, $\phi(G) \leq 5$. Thomassen [10] verified the *weak 3-flow conjecture* by proving that every 8-edge-connected graph G admits a modulo 3-orientation, that is, $\phi(G) \leq 3$. He also proved the weak version of Jaeger's circular flow conjecture. In [8] the edge-connectivity 8 was lowered to 6, and it was proved that every $6p$-edge-connected graph G admits a modulo $(2p+1)$-orientation, that is, $\phi(G) \leq 2 + \frac{1}{p}$.

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Recently, Jaeger's circular flow conjecture has been disproved for all p, except the particularly important cases $p = 1, 2$ [3]. Thus it is a great challenge to find the needed edge-connectivity between $4p$ and $6p$.

In this paper we investigate the possibility of getting strict inequalities for the flow index. It turns out that strongly connected orientations are central for this.

Theorem 1.1. *Let G be a connected graph and p be a positive integer. Then $\phi(G) < 2 + \frac{1}{p}$ if and only if G has a strongly connected modulo $(2p + 1)$-orientation.*

Theorem 3.1 in Section 3 generalizes Theorem 1.1 to flow indices other than $2 + \frac{1}{p}$.

Theorem 1.1 can be used to show that an upper bound on the flow index is sharp. Jaeger's observation above gives an upper bound of the form $2 + \frac{1}{p}$ for the flow index. Theorem 1.1 can then be used to prove that this upper bound is in fact sharp. We give examples of this application in Section 5.

The case $p = 1$ is perhaps particularly interesting as the inequality $\phi(G) < 3$ is equivalent to the statement that G has an orientation such that, for every edge-cut (A, B), the number of edges from A to B is less than twice the number of edges from B to A and greater than half the number of edges from B to A.

As every 12-edge-connected graph G has a modulo 5-orientation [8], it satisfies $\phi(G) \le 5/2 < 3$. We show that the inequality $\phi(G) < 3$ also holds when G is 8-edge-connected.

Theorem 1.2. *For every 8-edge-connected graph G, the flow index $\phi(G) < 3$.*

Theorem 1.2 is a corollary of Theorem 1.1 and a technical orientation result, Theorem 4.2 in Section 4.

Conjecture 1.3. *For every 6-edge-connected graph G, the flow index $\phi(G) < 3$.*

K_6 has only one modulo 3-orientation, up to isomorphism, and that is not strongly connected. Thus $\phi(K_6) = 3$. So the assumption on the edge-connectivity in Conjecture 1.3 cannot be relaxed. It may be possible, though, to replace 3 in the conclusion by 2.9999.

2. Preliminaries

We follow the notation and terminology of [13]. In particular, an *integer flow* of a graph G is a pair (D, f) where D is an orientation of G, and f is an integer valued function defined on the edges such that, at every vertex, the in-flow equals the out-flow. A *modulo k flow* (where k is a natural number) is defined analogously, except that we only require that the in-flow equals the out-flow modulo k.

2.1. The flow index

Lemma 2.1 (*Tutte [11], see also [12]*)**.** *Let (D, f) be a modulo k-flow of a graph G. Then G admits an integer k-flow (D, f') such that $f'(e) \equiv f(e) \pmod{k}$ for every edge e of G.*

Lemma 2.2 (*Hoffman [4], see also [5], [1] p.88, and Theorem 2.3.1 in [13]*)**.** *Let G be a connected bridgeless graph, D be an orientation of G and a, b be two positive integers ($a \le b$). The following statements are equivalent.*

(i)

$$\frac{a}{b} \le \frac{|[A, B]_D|}{|[B, A]_D|} \le \frac{b}{a}$$

for every edge-cut (A, B) of G;

(ii) G admits a positive integer flow (D, f) such that $a \le f(e) \le b$ for each $e \in E(G)$.

Definition 2.3. Let k, d be two positive integers with $k > 2d$. An integer flow (D, f) of a graph G is called a *circular $\frac{k}{d}$-flow* if $d \le |f(e)| \le k - d$ for every edge $e \in E(G)$.

Definition 2.4. The *flow index* $\phi(G)$ of a connected bridgeless graph G is the minimum rational number r such that the graph admits a circular r-flow.

The existence of this minimum r was established by Goddyn, Tarsi and Zhang [2], see Lemma 2.7. By Lemma 2.1, we have the following lemma.

Lemma 2.5. *Let k, d be two positive integers and G be a graph. The graph G admits an integer flow (D, f) with $d \leq |f(e)| \leq k - d$ for every edge $e \in E(G)$ if and only if G admits a modulo k-flow (D, f') with $d \leq |f'(e)| \leq k - d$ for every edge $e \in E(G)$.*

By Lemma 2.5, we can replace "integer flow" in Definition 2.3 by "modulo k-flow".

Lemma 2.6. *Let k, d be two positive integers with $k > 2d$ and G be a connected bridgeless graph. Then $\phi(G) \leq \frac{k}{d}$ if and only if the graph G admits a modulo k-flow (D, f) with $f(e) \in \{d, \ldots, k - d\}$.*

Lemma 2.7 (*Goddyn, Tarsi and Zhang [2]*)**.** *Let G be a connected bridgeless graph. Then*

$$\phi(G) = \min\{\theta(D) : \textit{ for all strongly connected orientations } D \textit{ of } G\},$$

where

$$\theta(D) = \max\{\frac{|[A, B]_D| + |[B, A]_D|}{|[B, A]_D|} : \textit{ for all edge cuts } (A, B) \textit{ of } G\}.$$

Thus $\phi(G) \geq 2$ with equality if and only if G has a balanced orientation, that is, G is Eulerian. For a non-Eulerian graph G, $\phi(G)$ is close to 2 if and only if G has an orientation which is close to being balanced.

2.2. Strongly connected orientations

Strong connectedness of a digraph is clearly preserved under contraction. The following useful fact concerning contraction and strongly connectedness is also straightforward.

Lemma 2.8. *Let G be a graph with an orientation D.*

(a) D is strongly connected if and only if G is connected and every edge in D is contained in a directed cycle.

(b) If e is an edge which is contained in a directed cycle of D, then D is strongly connected if and only if D/e (that is, the graph obtained by contracting e) is strongly connected.

Lemma 2.9. *Let G be a bridgeless graph and $e = xy$ be an edge of G. If G/e has a strongly connected orientation D', then D' can be extended to a strongly connected orientation D of G.*

Proof. Let D'' be the orientation of $G - e$ given by D'. If D'' is strongly connected, then we give e any orientation. So assume that D'' is not strongly connected. As $G - e$ is connected, it follows from Lemma 2.8(a) that D'' has an edge e' which is not contained in a directed cycle in D''. As D' is strongly connected, it has a directed cycle C containing e'. As C is not a directed cycle in D'', the edge set of C forms a directed path from y to x, say. We direct e from x to y, and now the edge set of C (in D'') together with e form a directed cycle. The resulting orientation D' of G is strongly connected by Lemma 2.8(b). ∎

3. Theorem 1.1 and its generalization

Theorem 1.1 is a corollary of the following theorem when $k = 2p + 1$.

Theorem 3.1. *Let G be a connected bridgeless graph and, p, k be two positive integers with $k \geq 2p + 1$. Then the following statements are equivalent.*

(a) $\phi(G) < \frac{k}{p}$.

(b) G admits a modulo k-flow (D, f) with $f : E(G) \to \{p, \dots, k - p - 1\}$ such that D is strongly connected.

The proof of "(a) ⇒ (b)". By Lemma 2.7, there is an orientation D' of G such that

$$\frac{p}{k-p} < \frac{|[A, B]_{D'}|}{|[B, A]_{D'}|} < \frac{k-p}{p}, \tag{1}$$

for every edge-cut (A, B) of G.

By Lemma 2.2 ((i) ⇒ (ii)), let (D', f') be a positive integer flow of G with $f'(e) \in \{p, \dots, k - p\}$.

From (D', f'), we define *a modulo k-flow (D, f) where $f : E(G) \to \{p, \dots, k - p - 1\}$ such that, for each edge $e \in E(G)$,*

(α) if $f'(e) = p$, then $f(e) = f'(e) = p$ and e has the same orientation in D and D';

(β) if $f'(e) = k - p$, then $f(e) = k - f'(e) = p$ and e has opposite orientation in D and D';

(γ) if $f'(e) = \mu \notin \{p, k-p\}$, then we have the choices: Either $f(e) = \mu$ and e has the same orientation in D and D', or $f(e) = k - \mu$ and e has opposite orientation in D and D'.

It is obvious that (D, f) is a modulo k-flow with $f : E(G) \to \{p, \dots, k - p - 1\}$ for any choice in (γ). We now show that, with appropriate choices in (γ), the orientation D is strongly connected. We prove this by induction on $|F|$, where

$F = \{e \in E(G) : p + 1 \leq f'(e) \leq k - p - 1\}$.

(I) First suppose $F = \emptyset$. Then for each edge $e \in E(G)$, either $f'(e) = p$ or $f'(e) = k - p$.

Suppose that D is not strongly connected. Then there is an edge-cut (A, B) of G such that

$$[B, A]_D = \emptyset.$$

That is, all edges of (A, B) are oriented from A to B under the orientation D.

So for each edge $e \in (A, B)$, $D'(e) \in [A, B]_{D'}$ if and only if $f'(e) = f(e) = p$, and $D'(e) \in [B, A]_{D'}$ if and only if $f'(e) = k - f(e) = k - p$.

Notice that (D', f') is a positive flow which is balanced on the edge-cut (A, B). Hence

$$(k-p)|[B, A]_{D'}| = \sum_{e \in [B,A]_{D'}} f'(e) = f'^{-}(A) = f'^{+}(A) = \sum_{e \in [A,B]_{D'}} f'(e) = p|[A, B]_{D'}|.$$

That is,

$$\frac{|[A, B]_{D'}|}{|[B, A]_{D'}|} = \frac{k-p}{p}.$$

This contradicts Inequality (1).

(II) Now, suppose $F \neq \emptyset$. Let $e \in F$ and by induction, there exists a modulo k-flow (D'', f'') of G/e satisfying $(\alpha), (\beta), (\gamma)$ such that D'' is strongly connected.

Then, by applying Lemma 2.9, the orientation D'' of G/e can be extended to a strongly connected orientation D of the entire graph G. We further assign a flow value to e with $f(e) = f'(e)$ if D is consistent with D', or $f(e) = k - f'(e)$ otherwise. ■

The proof of "(b) ⇒ (a)". Let (D, f) be a modulo k-flow with $f : E(G) \to \{p, \dots, k - p - 1\}$ such that D is strongly connected.

For any edge $e \in E(G)$, D has a directed circuit Q_e containing e since D is strongly connected. Let (D, f_e) be the flow of G such that f_e is 1 on the edges of Q_e and 0 on all other edges.

Let $m = |E(G)| + 1$. Consider a modulo (mk)-flow (D, f') with $f' = mf + \sum_{e \in E(G)} f_e$.

For each edge $e \in E(G)$, $p \leq f(e) \leq k - p - 1$ and

$$mp + 1 \leq mf(e) + f_e(e) \leq f'(e) \leq mf(e) + |E(G)| \leq mk - mp - 1.$$

Hence (D, f') is a modulo (mk)-flow with $f' : E(G) \to \{mp+1, \ldots, mk-mp-1\}$. By Lemma 2.6,

$$\phi(G) \le \frac{mk}{mp+1} < \frac{k}{p}.$$

This completes the proof of "(b) ⇒ (a)". ∎

To see that Theorem 3.1 implies Theorem 1.1 it is clear that the implication (a) ⇒ (b) implies the "only if" part of Theorem 1.1 when $k = 2p+1$. To obtain the "if" part from the implication (b) ⇒ (a) we need a strongly connected modulo $(2p+1)$-flow with all flow values equal to p. We obtain that flow from the strongly connected modulo $(2p+1)$-orientation by letting all flow values be p.

4. Theorem 1.2

For vertex subsets $U, W \subseteq V(G)$, let $[U, W]_G = \{uw \in E(G) : u \in U, w \in W\}$. When $U = \{u\}$ or $W = \{w\}$, we write $[u, W]_G$, $[U, w]_G$, $E_G(u, w) = [u, w]_G$ and $E_G(z_0) = [z_0, V(G) \setminus \{z_0\}]$, respectively. The subscript G may be omitted when G is understood from the context. Also, the union of two sets A, B is for convenience denoted $A + B$.

Definition 4.1. (i) A mapping $\beta : V(G) \mapsto \{0, 1, 2\}$ is called a Z_3-boundary of G if $\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod 3$.

(ii) Let β be a Z_3-boundary of G. An orientation D of G is called a β-orientation if $d^+_D(v) - d^-_D(v) \equiv \beta(v) \pmod 3$ for each vertex $v \in V(G)$.

Let G be a graph with a Z_3-boundary β. Denote $\mathcal{P}(V(G))$ to be the power set of $V(G)$. Define a mapping $\tau : \mathcal{P}(V(G)) \mapsto \{0, \pm 1, \pm 2, \pm 3\}$ as follows: for each vertex $x \in V(G)$,

$$\tau(x) \equiv \begin{cases} \beta(x) \pmod 3; \\ d(x) \pmod 2. \end{cases}$$

For each nonempty $A \subset V(G)$, let $\beta(A) \equiv \sum_{x \in A} \beta(x) \pmod 3 \in \{0, 1, 2\}$ and $d(A) = |[A, V(G) \setminus A]|$. Define $\tau(A)$ as

$$\tau(A) \equiv \begin{cases} \beta(A) \pmod 3; \\ d(A) \pmod 2. \end{cases} \tag{2}$$

Theorem 1.2 is a corollary of Theorem 4.2, which is a further refinement of the techniques developed in [10].

Theorem 4.2. *Let G be a graph with a Z_3 boundary β, $z_0 \in V(G)$ and D_{z_0} be a pre-orientation of $E(z_0)$. Assume that*

(i) $|V(G)| \ge 3$;

(ii) $d(z_0) \le 4 + |\tau(z_0)|$ and, under the orientation D_{z_0}, $d^+_{D_{z_0}}(z_0) - d^-_{D_{z_0}}(z_0) \equiv \beta(z_0) \pmod 3$;

(iii) $d(A) \ge 6 + |\tau(A)|$ for each nonempty vertex subset A not containing z_0 such that $|V(G) \setminus A| > 1$.

Then the pre-orientation D_{z_0} of $E(z_0)$ can be extended to an orientation D of G such that

(a) D is a β-orientation, that is, for each vertex $x \in V(G)$, $d^+_D(x) - d^-_D(x) \equiv \beta(x) \pmod 3$;

(b) under the orientation D, $G - z_0$ is strongly connected.

Theorem 4.2 is similar to Theorem 3.1 in [8] for $k = 3$. The main difference is that we have replaced 4 in (iii) by 6. With this stronger condition we can conclude that $G - z_0$ can be chosen to be strongly connected.

4.1. Properties of τ

Proposition 4.3. *Let G be a graph with a Z_3-boundary β, let A, B, C be subsets of $V(G)$, and let t be a positive integer.*

(i) (Lovász et al. [8]) If $d(A) \ge 2t$, then $d(A) \ge 2t - 2 + |\tau(A)|$.

(ii) If $\tau(A) + \tau(B) + \tau(C) \equiv 0 \pmod 6$, then $|\tau(A)| + |\tau(B)| \ge |\tau(C)|$.

(iii) If $A \cap B = \emptyset$, then $\tau(A + B) \equiv \tau(A) + \tau(B) \pmod 6$.

Proof. (ii) Since $\tau(A)+\tau(B) \equiv -\tau(C) \pmod 6$ and $\tau(C) \in \{0, \pm 1, \pm 2, \pm 3\}$ by definition, we have $|\tau(A)|+|\tau(B)| \geq |\tau(A)+\tau(B)| \geq |\tau(C)|$.

(iii) By the definition of τ (see Eq. (2)), we have

$$\tau(A+B) \equiv \beta(A+B) \equiv \sum_{x\in A}\beta(x)+\sum_{x\in B}\beta(x) \equiv \beta(A)+\beta(B) \equiv \tau(A)+\tau(B) \pmod 3,$$
$$\tau(A+B) \equiv d(A+B) \equiv d(A)+d(B)-2|[A,B]| \equiv \tau(A)+\tau(B) \pmod 2.$$

Thus $\tau(A+B) \equiv \tau(A)+\tau(B) \pmod 6$. ■

Note that, if $\{A, B, C\}$ is a partition of $V(G)$, then $\tau(A)+\tau(B)+\tau(C) \equiv 0 \pmod 6$ since $\tau(B)+\tau(C) \equiv \tau(A^c) \equiv -\tau(A) \pmod 6$ by Proposition 4.3(iii).

The following proposition plays an important role in the proof of Theorem 4.2.

Proposition 4.4. *Let G be a graph with a Z_3-boundary β, let $\{A, B, C\}$ be a partition of $V(G)$, and let a, b, c be positive integers. Assume that $d(A) \geq 2a+|\tau(A)|$, $d(B) \geq 2b+|\tau(B)|$ and $d(C) \leq 2c+|\tau(C)|$. Then*

(i) $|[A,B]| \geq a+b-c$.

(ii) If $|[A,B]| = a+b-c$, then all the inequalities above are equalities, and $|\tau(A)|+|\tau(B)| = |\tau(C)|$.

Proof. Since $\{A, B, C\}$ is a partition of $V(G)$, Proposition 4.3(iii) implies that $\tau(A)+\tau(B)+\tau(C) \equiv 0 \pmod 6$. Thus $|\tau(A)|+|\tau(B)| \geq |\tau(C)|$ by Proposition 4.3(ii). It follows that

$$|[A,B]| = \frac{d(A)+d(B)-d(C)}{2} \geq a+b-c+\frac{|\tau(A)|+|\tau(B)|-|\tau(C)|}{2} \geq a+b-c.$$

Hence (i) holds. To prove (ii) assume that $|[A,B]| = a+b-c$. Then all inequalities above are all equalities. In particular, $|\tau(A)|+|\tau(B)| = |\tau(C)|$. ■

If the triple $(A, B; C)$ satisfies the assumption of Proposition 4.4 for some positive natural numbers a, b, c and also satisfies (ii), we call it an *extreme triple*.

4.2. Proof of Theorem 4.2

The proof is by induction. Suppose (reductio ad absurdum) that the theorem is false, and let $\mathcal{M}$ be the collection of counterexamples (G, β, z_0) such that $|E(G-z_0)|$ is minimum.

When A is a vertex subset of a graph G', we shall use $d'(A)$, $\beta'(A)$ and $\tau'(A)$ for the corresponding notions in G'.

We shall establish a number of properties of all members of $\mathcal{M}$ that will lead to a contradiction.

Let (G, β, z_0) be any member of $\mathcal{M}$.

Claim 1. *$G-z_0$ is 4-edge-connected and $|V(G-z_0)| \geq 3$.*

Proof. Let $\{A, B\}$ be a partition of $G-z_0$. Then $\{A, B, z_0\}$ is a partition of $V(G)$. It follows by Proposition 4.4(i) that $|[A,B]| \geq 4$, and so $G-z_0$ is 4-edge-connected. If $|V(G-z_0)| = 2$, denote $V(G-z_0) = \{x, y\}$. Then $|E(x,y)| \geq 4$. Orient one edge in $E(x,y)$ from x to y, another edge from y to x, and then orient the remaining edges to modify the boundary β. This results a β-orientation of G such that $G-z_0$ is strongly connected. ■

Note that two parallel edges are enough to modify the Z_3-boundary of the end vertices.

Claim 2. *For any $x, y \in V(G)-z_0$, $|E(x,y)| \leq 2$.*

Proof. Suppose that $|E(x,y)| \geq 3$ and let $e_1, e_2 \in E(x,y)$. We first apply induction on the contracted graph $G' = G/E(x,y)$ with a modified boundary β', where $\beta'(v) = \beta(v)$ for any $v \in V(G)-x-y$, and $\beta'(w) \equiv \beta(x)+\beta(y) \pmod 3$ for the contracted vertex w. By Lemma 2.9, the strongly connected orientation of $G'-z_0 = G/E(x,y)-z_0$ can be extended to a strongly connected orientation D'' of $G-z_0-e_1-e_2$. Note that $G-z_0-e_1-e_2$ is bridgeless by Claim 1. Then we add e_1, e_2 back and orient them appropriately to modify the boundary $\beta(x)$, $\beta(y)$. This results a β-orientation D of G. $G-z_0$ is strongly connected under orientation D since $D''(G-z_0-e_1-e_2)$ is strongly connected. ■

In a series of Subclaims 3.1–3.7, we aim to show the following major part of the proof.

Claim 3. *$d(A) \geq 8 + |\tau(A)|$ for any $A \subset V(G) - z_0$ with $1 < |A| < |V(G) - z_0|$.*

For a set $A \subset V(G) - z_0$ with $1 < |A| < |V(G) - z_0|$, we call A a **critical subset** if $d(A) = 6 + |\tau(A)|$ and, for any $A' \subset A$ with $1 < |A'| < |A|$, $d(A') \geq 8 + |\tau(A')|$. Suppose that Claim 3 fails and critical subset exists.

For a **critical subset** A, let $B = V(G) - A - z_0$, $K(A) = \{x \in A : N_G(x) \cap B \neq \emptyset\}$ and $k(A) = |K(A)|$. We further denote $K(A) = \{x_1, \ldots, x_{k(A)}\}$. The following properties, stated as Subclaims 3.1–3.4, hold for **any critical subset** A.

Subclaim 3.1. $|A| \geq 3$.

Proof. Assume not and denote $A = \{x, y\}$. Since $d(A) = d(A^c) = 6 + |\tau(A)|$ and applying Proposition 4.4(i) for the partition $\{x, y, A^c\}$, we have $|E(x, y)| \geq 3$, contradicting Claim 2. ■

Let $G_1 = G/A$. By induction, there is a β-orientation D_1 in G_1 whose restriction to $G_1 - z_0$ is strongly connected. As $D_1(G_1 - z_0)$ is strongly connected, there exist both edges oriented towards A and away from A.

With a slight abuse of notation, we also let G denote the partially oriented graph (with the partial orientation D_1) such that all edges in $G[A]$ are undirected and all edges not in $G[A]$ are directed.

Subclaim 3.2. *For $i \neq j$, there is no directed path from x_i to x_j under the orientation $D_1(G - z_0)$.*

Proof. Suppose, without loss of generality, that $D_1(G - z_0)$ contains a directed path P from x_1 to x_2. We contract A^c to a new z_0, delete an edge from x_1 to z_0 and another one from z_0 to x_2 and add a new undirected edge x_1x_2. If the resulting graph G' satisfies conditions (i)(ii)(iii) of Theorem 4.2, then by the minimality of (G, β, z_0), we can apply induction on G' with a modified boundary β' to find a strongly connected orientation $D'(G' - z_0)$. We may further assume the new added edge x_1x_2 is oriented from x_1 to x_2 in $D'(G' - z_0)$. Otherwise, we reverse a directed cycle containing x_2x_1 in $D'(G' - z_0)$ and the resulting orientation is still strongly connected in $G' - z_0$, and it preserves the boundary β'. Then we combine the orientations D_1 and D' to result an orientation D of G by deleting the new added edge x_1x_2. Now, under the orientation D, $G'' = G[A] \cup E(P)$ is strongly connected (as $D'(G' - z_0)$ is strongly connected), and $G - z_0/G''$ is strongly connected. By Lemma 2.8, $G - z_0$ is strongly connected under the orientation D, yielding a contradiction to $(G, \beta, z_0) \in \mathcal{M}$.

To show the induction is possible for G', it suffices to verify conditions (ii)(iii) of Theorem 4.2. Clearly, condition (iii) is satisfied for all singletons. Since A is critical, for any $A' \subset A$ with $1 < |A'| < |A|$, $d(A') \geq 8 + |\tau(A')|$ in G. Thus $d'(A') \geq d(A') - 2 \geq 6 + |\tau(A')| = 6 + |\tau'(A')|$ in G'. Moreover, the new z_0 of G' satisfies $d'(z_0) = d(A) - 2 = 4 + |\tau(A)| = 4 + |\tau'(z_0)|$. Hence conditions (i)–(iii) are verified for G'. ■

For $i = 1, 2, \ldots, k(A)$, under the orientation $D_1(G - z_0)$, let
$Q_i^+ = \{x \in B :$ *there is a directed path in $D_1(G - z_0)$ from x_i to x*$\}$ and
$Q_i^- = \{x \in B :$ *there is a directed path in $D_1(G - z_0)$ from x to x_i*$\}$.

Subclaim 3.3. *Each of the following holds.*

(a) $d^+_{D_1}(x_i) \geq 1$ and $d^-_{D_1}(x_i) \geq 1$ for $i = 1, 2, \ldots, k(A)$.

(b) $Q_i^+ = Q_i^- = Q_i$ for $i = 1, 2, \ldots, k(A)$, and $\{Q_1, Q_2, \ldots, Q_{k(A)}\}$ is a partition of B.

(c) For $i \neq j$, $[Q_i, Q_j] = \emptyset$.

(d) $k(A) \leq 2$.

(e) $d(x_i) = 6 + |\tau(x_i)| \geq 8$ for $i = 1, \ldots, k(A)$.

(f) For $i = 1, \ldots, k(A)$, $|Q_i| \geq 2$, and either $[Q_i, A + B - Q_i]$ or $[Q_i + x_i, A + B - Q_i - x_i]$ is a 4-edge-cut in $G - z_0$.

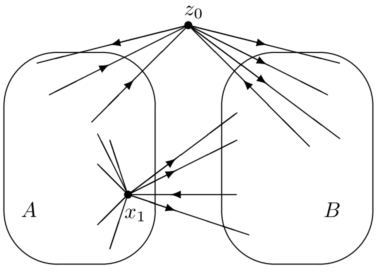

Fig. 1. Structure of G in Subclaim 3.4.

Proof. (a) Suppose $d^+_{D_1}(x_i) = 0$ and let ux_i be a directed edge from B to A. Since $D_1(G_1 - z_0)$ is strongly connected, there is a directed path from the contracted vertex to u, and this yields a directed path P from x_j to u under the orientation $D_1(G - z_0)$. By the assumption of $d^+_{D_1}(x_i) = 0$, we have $x_i \neq x_j$. This results a directed path $P + ux_i$ from x_j to x_i in $D_1(G - z_0)$, a contradiction to Subclaim 3.2.

(b) and (c) follow by the definitions of Q_i^+, Q_i^- and Subclaim 3.2.

(d) If $k(A) \geq 3$, by (c), Claim 1 and Proposition 4.3(i), we have

$$d(A) \geq \sum_{i=1}^{3} |[x_i, Q_i]| \geq 12 \geq 9 + |\tau(A)| > 6 + |\tau(A)|,$$

contradicting that A is critical.

(e) By Claim 1 and Subclaim 3.3(c), $|[x_i, Q_i]| \geq 4$ and $|[x_i, A - x_i]| \geq 4$. Hence $d(x_i) \geq 8$.

If $d(x_i) \geq 8 + |\tau(x_i)|$ for some i, we contract A^c to a new z_0 and delete a pair of edges with opposite directions (by (a)) between x_i and the new z_0. We verify conditions of Theorem 4.2 in order to apply the induction on the resulting graph G' with the modified boundary β'. Since $d(x_i) \geq 8 + |\tau(x_i)|$, we have $d'(x_i) = d(x_i) - 2 \geq 6 + |\tau(x_i)| = 6 + |\tau'(x_i)|$, and so condition (iii) is satisfied for x_i, as well as for other singletons. Since A is critical, $d'(A') \geq d(A') - 2 \geq 6 + |\tau'(A')|$ for any $A' \subset A$ with $1 < |A'| < |A|$ in G'. The new z_0 of G' also satisfies $d'(z_0) = 4 + |\tau'(z_0)|$ since $d(A^c) = d(A) = 6 + |\tau(A)|$. Now, we can apply induction to G'. By induction, there is a β'-orientation D' of G' such that $D'(G' - z_0)$ is strongly connected. We combine the orientations D' and D_1 to get an orientation D of G. Then D is a β-orientation of G and $D(G - z_0)$ is strongly connected by Lemma 2.8. This proves (e).

(f) By Claim 1 and Subclaim 3.3(c), $|[x_i, Q_i]| \geq 4$. By Claim 2, $|Q_i| \geq 2$. By (e), $d(x_i) \leq 9$. Also,

$$\begin{aligned} &|[Q_i, A + B - Q_i]| + |[Q_i + x_i, A + B - Q_i - x_i]| + |[x_i, z_0]| \\ &= |[x_i, Q_i]| + |[x_i, A - x_i]| + |[x_i, z_0]| = d(x_i) \leq 9, \end{aligned}$$

so either $|[Q_i, A + B - Q_i]| = 4$ or $|[Q_i + x_i, A + B - Q_i - x_i]| = 4$. ■

Subclaim 3.4. $k(A) = 2$.

Proof. By Subclaim 3.3(d), suppose to the contrary that $k(A) = 1$ and $Q_1 = B$, see Fig. 1. Since $G - z_0$ is 4-edge-connected by Claim 1, we have

$$|[x_1, B]| = |[A, B]| \geq 4. \tag{3}$$

As A is critical and $|A| \geq 3$ by Subclaim 3.1, it follows that $d(A - x_1) \geq 8 + |\tau(A - x_1)|$. By applying Proposition 4.4(i) (with $(a, b, c) = (4, 3, 2)$) to the partition $\{A - x_1, B + x_1, z_0\}$, we have

$$|[x_1, A - x_1]| = |[A - x_1, B + x_1]| \geq 5. \tag{4}$$

It follows by Eqs. (3)(4) and Subclaim 3.3(e) that

$$9 \geq 6 + |\tau(x_1)| = d(x_1) = |[x_1, A - x_1]| + |[x_1, B]| + |E(x_1, z_0)| \geq 5 + 4 + |E(x_1, z_0)|.$$

Hence, all inequalities are equalities, that is,

$$9 = d(x_1) = 6 + |\tau(x_1)|,$$

$$|\tau(x_1)| = 3, \tag{5}$$

$$|E(x_1, z_0)| = 0, \tag{6}$$

$$|[x_1, B]| = |[A, B]| = 4, \tag{7}$$

and

$$|[A - x_1, B + x_1]| = |[x_1, A - x_1]| = 5. \tag{8}$$

Thus, by Eq. (8), $(A - x_1, B + x_1; z_0)$ is an extreme triple with

$$d(A - x_1) = 8 + |\tau(A - x_1)|. \tag{9}$$

Thus, we have

$$\begin{aligned} 6 + |\tau(A)| &= d(A) = |[x_1, B]| + |[z_0, A]| \\ &= 4 + |[z_0, A - x_1]| + |E(x_1, z_0)| \quad \text{(by Eq. (7))} \\ &= 4 + (d(A - x_1) - |[x_1, A - x_1]|) \quad \text{(by Eq. (6))} \\ &= 7 + |\tau(A - x_1)| \quad \text{(by Eqs. (9) and (8))}. \end{aligned}$$

That is,

$$|\tau(A)| = 1 + |\tau(A - x_1)|. \tag{10}$$

Since $|\tau(x_1)| = 3$ by Eq. (5), then by applying Proposition 4.3(iii), we have

$$\tau(A - x_1) \equiv \tau(A) - \tau(x_1) \equiv \tau(A) - 3 \pmod{6}. \tag{11}$$

By examining all possible values of $\tau(A)$ in Eqs. (10) and (11), we conclude that $\tau(A) \in \{\pm 2\}$.

We contract A^c to a new z_0. Since $\tau(A) \in \{\pm 2\}$ and the new z_0 satisfies $\tau(z_0) = \tau(A^c)$, we have $\tau(z_0) \in \{\pm 2\}$ in G/A^c. If $\tau(z_0) = 2$, we delete an edge oriented from x_1 to the new z_0, and then decrease $\beta(x_1)$ by 1 and increase $\beta(z_0)$ by 1; if $\tau(z_0) = -2$, we delete an edge oriented from the new z_0 to x_1, and then increase $\beta(x_1)$ by 1 and decrease $\beta(z_0)$ by 1. In the resulting graph G' with modified boundary β', we have $|\tau'(z_0)| = |\tau(z_0)| + 1 = 3$ since $\tau'(z_0) = \tau(z_0) + 1 = 3$ in the former case and $\tau'(z_0) = \tau(z_0) - 1 = -3$ in the latter case. That is, $d'(z_0) = d(A) - 1 = 7 = 4 + |\tau'(z_0)|$, and so condition (ii) of Theorem 4.2 is satisfied for z_0. As $d'(x_1) = d(x_1) - 1 = 8$, then by Proposition 4.3(i), $d'(x_1) \geq 6 + |\tau'(x_1)|$. Hence condition (iii) is satisfied for x_1, as well as all singletons.

For an $A' \subset A$ with $1 < |A'| < |A|$ in G', if $d'(A') = d(A') - 1$, then $|\tau'(A')| = |\tau(A') \pm 1| \leq |\tau(A')| + 1$, and since A is critical, $d'(A') = d(A') - 1 \geq 8 + |\tau(A')| - 1 \geq 6 + |\tau'(A')|$; if $d'(A') = d(A')$, then $|\tau'(A')| = |\tau(A')|$ and $d'(A') = d(A') \geq 8 + |\tau(A')| = 8 + |\tau'(A')|$ by A being critical. Hence conditions of Theorem 4.2 are satisfied for G'. By the induction, the resulting orientation D' of G' together with D_1 yields a β-orientation D of G such that $D(G - z_0)$ is strongly connected by Lemma 2.8. This contradicts $(G, \beta, z_0) \in \mathcal{M}$ and proves Subclaim 3.4. ■

Subclaim 3.5. *Let $\{U, W\}$ be a partition of $G - z_0$ with $|U| \geq 2$ and $|W| \geq 2$. If $[U, W]$ is a 4-edge-cut in $G - z_0$, then both U and W contain a critical subset.*

Proof. If $|[U, W]| = 4$, then, by Proposition 4.4(ii), $(A, B; z_0)$ is an extreme triple with

$$d(U) = 6 + |\tau(U)| \text{ and } d(W) = 6 + |\tau(W)|.$$

It follows by the definition of critical subset that both U and W contain critical subset. (Possibly U or W itself is a critical subset.) ■

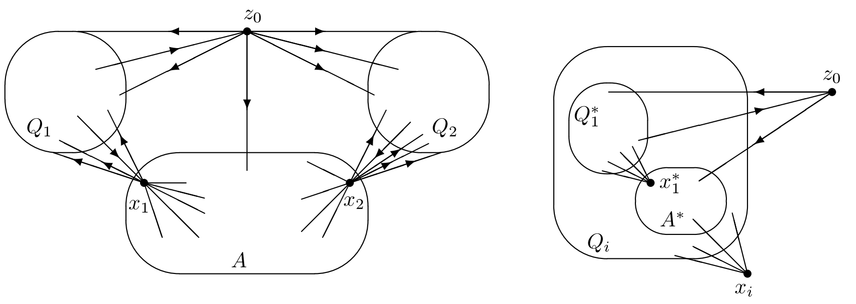

Fig. 2. Structure of critical subset A.

As a consequence of Subclaims 3.3, 3.4, and 3.5, we have the following structural description of $G - z_0$ for every critical subset A. (Let x_1, x_2, Q_1, Q_2 be defined as in Subclaim 3.3.)

Subclaim 3.6. *For every $i \in \{1, 2\}$, each of the following holds (see Fig. 2).*
(i) x_i is a cut-vertex of $G - z_0$ separating $(Q_{3-i} + A - x_i)$ and Q_i;
(ii) There is a 4-edge-cut of $G - z_0$ which is either $[x_i, Q_i]$ or $[x_i, A - x_i]$.
(iii) There is a critical subset A^* which is contained in $Q_i + x_i$.
(iv) Let x_1^*, x_2^*, Q_1^* and Q_2^* be corresponding notions defined in Subclaim 3.3 with respect to the critical subset A^*. Then one of $\{Q_1^*, Q_2^*\}$ is contained in $Q_i - A^*$.

Proof. Note that, by Subclaim 3.4, $k(A) = 2$ for every critical subset A.
(i) and (ii) are immediate corollaries of the combination of Subclaim 3.3(b), (c) and (f).
By (ii) and Subclaim 3.5, $Q_1 + x_1$ contains a critical subset. Thus (iii) holds.
(iv) Notice that $(Q_1^* + x_1^*) \cap (Q_2^* + x_2^*) = \emptyset$ by Subclaim 3.3(b). We may, without loss of generality, assume $x_i \notin Q_1^* + x_1^*$. Then we have $x_1^* \in A^* - x_i \subset Q_i$ and all the neighbors of x_1^* are in $Q_i + x_i + z_0$. Moreover, if $x_i \notin A^*$, then x_i is not adjacent to x_1^* by the definition of Q_1^*. Thus $Q_1^* \cap (A + Q_{3-i}) = \emptyset$ since $G[Q_1^*]$ is connected (because $G - z_0$ is 4-edge-connected by Claim 1, x_1^* is a cutvertex of $G - z_0$ by Subclaim 3.6(i), and x_1^* has degree at most 9 by Subclaim 3.3(e)). Therefore, Q_1^* is contained in $Q_i - A^*$. ■

Subclaim 3.7. *There is no critical subset, and therefore, Claim 3 holds.*

Proof. Assume critical subset exists. For any critical subset A, define $w(A) = \min\{|Q_1|, |Q_2|\}$. Choose a critical subset A such that $w(A)$ minimized among all possible choices. We assume, without loss of generality, that $w(A) = \min\{|Q_1|, |Q_2|\} = |Q_1|$. By Subclaim 3.6(iii) and (iv), there is another critical subset A^* contained in $Q_1 + x_1$ such that one of $\{Q_1^*, Q_2^*\}$ is contained $Q_1 - A^*$. It follows by Subclaim 3.1 that

$$w(A^*) = \min\{|Q_1^*|, |Q_2^*|\} \le |Q_1 - A^*| \le |Q_1| - 2 < |Q_1| = w(A),$$

a contradiction to the choice of A.
This contradiction implies there is no critical subset and establishes Claim 3. ■

Claim 4. *For any vertex $x \in V(G - z_0)$, x has at least three neighbors in G and*

$$d(x) = 6 + |\tau(x)|.$$

Proof. If x is not a neighbor of z_0, then x has at least three neighbors since $d(x) \ge 6$ and by Claim 2. If x is a neighbor of z_0, by Claims 1 and 2, x has at least two neighbors in $G - z_0$.
Suppose that $d(x) \ge 8 + |\tau(x)|$ for some $x \in V(G - z_0)$. Let y, z be distinct neighbors of x in $V(G - z_0)$. Delete xy and xz and add yz. We apply the induction to the resulting graph G'. D_{z_0} can be extended to

D' of G'. This results a β-orientation D of G such that $D(G-z_0)$ is strongly connected, a contradiction. To show that induction is possible for G', we first observe that condition (iii) is satisfied for singletons. By Claim 3 and $d'(A) \geq d(A) - 2$, condition (iii) holds for other subset A with $1 < |A| < |V(G') - z_0|$ as well. ■

Claim 5. *$\tau(x) \neq 0$ for any vertex $x \in V(G - z_0)$.*

Proof. Suppose $\tau(x) = 0$ for some vertex x other than z_0. Then $d(x) = 6$ and x has at least three neighbors by Claim 4. Since $G - z_0$ is 4-edge-connected (by Claim 1) and $d(x) = 6$, we have $|E(x, z_0)| \leq d(x) - 4 = 2$. We completely lift the edges incident with x, that is, we delete x and replace its six incident edges by three edges. Note that this lifting is possible by Claims 1, 2 and 4. We then apply the induction to the resulting graph G'. Since $|E(x, z_0)| \leq 2$, at least one pair of lifted edges is contained in $E(G - z_0)$, hence the strongly connected orientation $D'(G' - z_0)$ also results in a strongly connected orientation $D(G - z_0)$. To see that induction is possible, it suffices to verify condition (iii) for G' with the corresponding boundary β'. Clearly, condition (iii) holds for all singletons. For an $A' \subset V(G') - z_0$ with $1 < |A'| < |V(G') - z_0|$, we consider the sets $A', A' + x$ in G. As both $d(A') \geq 8 + |\tau(A')|$ and $d(A' + x) \geq 8 + |\tau(A' + x)|$ by Claim 3, we have

$$d'(A') \geq \frac{d(A') + d(A' + x) - d(x)}{2} \geq 5 + \frac{|\tau(A')| + |\tau(A' + x)|}{2} = 5 + |\tau'(A')|.$$

The last equality follows from Proposition 4.3(iii). Thus $d'(A') \geq 6 + |\tau'(A')|$ follows by parity, more precisely, Eq. (2). ■

Claim 6. *$\tau(x)\tau(y) > 0$ for any $x, y \in V(G) - z_0$.*

Proof. Suppose $\tau(x)\tau(y) \leq 0$. By Claim 5, we may assume $\tau(x) > 0$ and $\tau(y) < 0$. By Claim 1, there is a path joining x and y in $G - z_0$. So there exists an edge x_1y_1 of the path such that $\tau(x_1) > 0$ and $\tau(y_1) < 0$. We delete x_1y_1, decrease $\beta(x_1)$ by 1 and increase $\beta(y_1)$ by 1. Let $G' = G - x_1y_1$ be the resulting graph with the modified boundary β'. If G' and β' satisfy the conditions of Theorem 4.2, then by the definition of $\mathcal{M}$ and $|E(G' - z_0)| < |E(G - z_0)|$, we obtain a β'-orientation D' of G' such that $D'(G' - z_0)$ is strongly connected. D' can be modified to an orientation D of G by adding a directed edge from x_1 to y_1, yielding a contradiction. Hence, it suffices to verify condition (iii) for the vertices x_1, y_1 and each vertex subset A which are affected by deletion of x_1y_1.

Since $0 \leq \tau'(x_1) = \tau(x_1) - 1$ and $0 \geq \tau'(y_1) = \tau(y_1) + 1$, we have $|\tau'(x_1)| = |\tau(x_1)| - 1$ and $|\tau'(y_1)| = |\tau(y_1)| - 1$. Thus condition (iii) is satisfied for x_1, y_1.

For an $A \subset V(G') - z_0$ with $1 < |A| < |V(G') - z_0|$ in G', if $d'(A) = d(A) - 1$, then $|\tau'(A)| = |\tau(A) \pm 1| \leq |\tau(A)| + 1$, and so $d'(A) = d(A) - 1 \geq 6 + |\tau'(A)|$ by Claim 3; if $d'(A) = d(A)$, then either each of x_1, x_2 or none of x_1, x_2 is contained in A. In either case we have $|\tau'(A)| = |\tau(A)|$, and thus $d'(A) = d(A) \geq 8 + |\tau'(A)|$ by Claim 3. Hence condition (iii) holds for G'. ■

Let $V^+ = \{x \in V(G) - z_0 : \tau(x) = 1 \text{ or } 2\}$ and $V^- = \{x \in V(G) - z_0 : \tau(x) = -1 \text{ or } -2\}$.

Claim 7. *$V(G) - z_0 = V^+$ or $V(G) - z_0 = V^-$.*

Proof. By Claim 6, it suffices to show $|\tau(x)| \neq 3$ for any $x \in V(G) - z_0$. If $|\tau(x)| = 3$, then for a neighbor y of x in $G - z_0$, we can choose $\tau(x) = 3$ or $\tau(x) = -3$ so that $\tau(x)\tau(y) \leq 0$, yielding a contradiction to Claim 6. ■

Claim 8. *$d(z_0) = 4 + |\tau(z_0)|$.*

Proof. Suppose $d(z_0) \leq 2 + |\tau(z_0)|$. We obtain a graph G' by subdividing an edge $xy \in E(G - z_0)$ with an internal vertex z_0', identifying z_0' with z_0, and then orienting xz_0 from x to z_0 and yz_0 from z_0 to y. The resulting graph G' with the boundary $\beta' = \beta$ satisfies condition (ii) since $d'(z_0) = d(z_0) + 2 \leq 4 + |\tau(z_0)|$. For any vertex subset $A \subset V(G') - z_0$ with $1 < |A| < |V(G') - z_0|$ in G', $d'(A) = d(A) + 2$ if A contains both x and y, and $d'(A) = d(A)$ otherwise. So condition (iii) is satisfied.

Since $|E(G'-z_0)| < |E(G-z_0)|$ and by the definition of $\mathcal{M}$, an extension of D_{z_0} exists in G', resulting in an orientation of G, a contradiction. ■

Claims 1–8 hold for any member in $\mathcal{M}$. The following Claim 9 shows a further property for a member (G, β, z_0) of $\mathcal{M}$ such that $|E(G)|$ is minimized.

Claim 9. *Let (G, β, z_0) be a member of $\mathcal{M}$ such that $|E(G)|$ is minimized. If $V(G) - z_0 = V^+$ (V^-, respectively), then all edges incident with z_0 are directed away from z_0 (towards z_0, respectively).*

Proof. Suppose $V(G) - z_0 = V^+$ and xz_0 is directed towards z_0. We delete xz_0, decrease $\beta(x)$ by 1 and increase $\beta(z_0)$ by 1. In the resulting graph G', since $|\tau'(z_0)| = |\tau(z_0) \pm 1| \le |\tau(z_0)| + 1$, we have $d'(z_0) = d(z_0) - 1 \le 4 + |\tau(z_0)| - 1 \le 4 + |\tau'(z_0)|$. This verifies condition (ii) for G'. By a similar argument as in Claim 6, condition (iii) is satisfied as well. Since $|E(G)|$ is minimized in $\mathcal{M}$ and $|E(G')| < |E(G)|$, the pre-orientation of G' can be extended to a β'-orientation D' of G', and then to a β-orientation D of G by adding the directed edge xz_0. Moreover, $D'(G' - z_0)$ is strongly connected, which results a strongly connected orientation $D(G - z_0)$. This contradicts (G, β, z_0) being a member of $\mathcal{M}$. ■

The final step. Let (G, β, z_0) be a member of $\mathcal{M}$ such that $|E(G)|$ is minimized. Without loss of generality, assume that $V(G) - z_0 = V^+$. For if $V(G) - z_0 = V^-$, we replace $\beta(x)$ by $3 - \beta(x)$ for each vertex $x \in V(G)$ and reverse the directions of all edges incident with z_0. The resulting graph with modified boundary satisfies $V(G) - z_0 = V^+$ and $|E(G)|$ is minimized.

By Claims 8 and 9, we have $d(z_0) = 4 + |\tau(z_0)|$ and $d(z_0) \equiv \beta(z_0) \equiv \tau(z_0) \pmod 3$. Hence $\tau(z_0) = -1$ and $d(z_0) = 5$. Let xz_0 be an edge directed from z_0 to x. We obtain a graph G' by replacing xz_0 with a pair of parallel directed edges from x to z_0 and let the boundary $\beta' = \beta$. We shall show that G' with the boundary β' satisfies conditions of Theorem 4.2 and, furthermore, $\tau'(x) \in \{-1, -2\}$ for the vertex x.

Since $d'(z_0) = 6$ and $\tau'(z_0) = 2$, condition (ii) is satisfied. As $d'(x) = d(x) + 1 = 6 + |\tau(x)| + 1 \ge 8$ by Claim 5, we have $d'(x) \ge 6 + |\tau'(x)|$ by Proposition 4.3(i). Thus condition (iii) is satisfied for x. For any $A \subset V(G') - z_0$ with $1 < |A| < |V(G') - z_0|$ in G', if A contains x, then by Claim 3, we have $d'(A) \ge d(A) + 1 \ge 8 + 1 \ge 6 + |\tau'(A)|$; if A does not contain x, then $d'(A) = d(A) \ge 8 + |\tau'(A)|$. So conditions of Theorem 4.2 are satisfied for G' with the boundary β'.

As $V(G) - z_0 = V^+$, we have $\tau(x) \in \{1, 2\}$. Since $\tau'(x) \equiv \beta(x) \equiv \tau(x) \pmod 3$ and $\tau'(x) \equiv d'(x) \equiv \tau(x) + 1 \pmod 2$, we conclude that $\tau'(x) \in \{-1, -2\}$.

Now, since $\tau'(x) \in \{-1, -2\}$ and $V'^+ = V(G') - z_0 - x \ne \emptyset$ by Claim 1, (G', β', z_0) is not a member of $\mathcal{M}$ by Claim 7. By the definition of $\mathcal{M}$ and the fact that $|E(G' - z_0)| = |E(G - z_0)|$, there exists a β'-orientation D' of G' such that $D'(G' - z_0)$ is strongly connected. This results a β-orientation D of G by replacing the two edges from x to z_0 with one edge in opposite direction. The orientation D satisfies the theorem since $D(G - z_0) = D'(G' - z_0)$ is strongly connected, a contradiction to $(G, \beta, z_0) \in \mathcal{M}$. This completes the proof of Theorem 4.2. ■

4.3. *Proof of Theorem 1.2*

Let G be an 8-edge-connected graph. Construct a new graph G^+ by adding a new isolated vertex z_0 and a boundary function $\beta : V(G^+) \to \{0\}$. In order to apply Theorem 4.2, it is sufficient to prove that all conditions of Theorem 4.2 are satisfied for G^+.

(i) and (ii) are trivial.

For (iii), let $A \subset V(G)$ with $1 \le |A| < |V(G)|$. Note that $G^+ - z_0 = G$ is 8-edge-connected. Thus $d(A) \ge 8$. Hence $d(A) \ge 6 + |\tau(A)|$ by Proposition 4.3(i). ■

With a similar argument as in Theorem 4.12 of [8], Theorem 1.2 also holds under the weaker condition that each odd edge-cut has at least 9 edges.

5. Application to contractible configurations and computation of the flow index

A graph H is a *contractible configuration* for a graph property $\mathcal{P}$ if, for every supergraph G containing H as a subgraph, G/H has the property $\mathcal{P}$ if and only if G has the property $\mathcal{P}$. Jaeger et al. [7] introduced

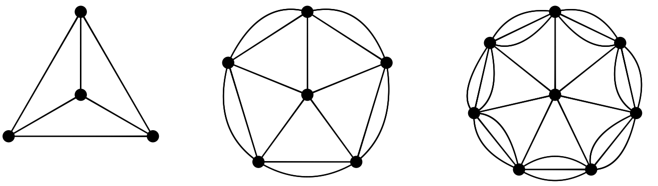

Fig. 3. Graph family $W(k) = kC_{2k+1} \cdot K_1$ for $k = 1, 2, 3$.

the concept of *group connectivity* which is useful for contractible configurations in connection with nowhere-zero flows. Jaeger et al. [7] showed that every 3-edge-connected graph is Z_6-connected, and every 4-edge-connected graph is Z_4-connected. That is, every 4-edge-connected graph is a contractible configuration for the graph property $\phi \leq 4$. Lovász et al. [8] showed that every 6-edge-connected graph is a contractible configuration for the graph property $\phi \leq 3$. Theorem 4.2 shows that every 8-edge-connected graph is a contractible configuration for the graph property $\phi < 3$.

Theorem 5.1. *Let H be an 8-edge-connected graph. Then, for every supergraph G of H,*

$$\phi(G) < 3 \text{ if and only if } \phi(G/H) < 3.$$

We conclude with a remark on the computation of $\phi(G)$ for an infinite class of non-Eulerian graphs.

Steffen [9] proved that, for every $k \geq 1$, the flow index of the complete graph K_{2k+2} is precisely $2+\frac{2}{k}$. We shall here give a short argument when k is even, that is $k = 2p$. We consider two disjoint copies of K_{2p+1} in K_{4p+2}. In each of them we orient the edges so that each vertex has indegree and outdegree p. Then we direct all edges from one of the complete subgraphs to the other. Then every vertex has indegree and outdegree p or $3p + 1$. This is a modulo $(2p + 1)$-orientation, so, by the observation of Jaeger [6] mentioned in the introduction, G has flow index at most $2 + \frac{1}{p}$. Clearly, every modulo $(2p + 1)$-orientation must have this structure, and since this orientation is not strongly connected, it follows from Theorem 1.1 that K_{4p+2} has flow index precisely $2 + \frac{1}{p}$.

Using this method, we can give other examples of $(2k+1)$-regular, $(2k+1)$-edge-connected graphs with flow index precisely $2 + \frac{2}{k}$. Consider the planar graph $W(k) = kC_{2k+1} \cdot K_1$ in Fig. 3. Assume again that $k = 2p$. One can show that $W(2p)$ has a modulo $(2p + 1)$-orientation. (The argument is slightly tedious but straightforward, so we leave it for the reader.) Hence $W(2p)$ has flow index at most $2 + \frac{1}{p}$. To prove that this bound is sharp, consider any modulo $(2p+1)$-orientation of $W(2p)$. Half the vertices have outdegree p, and half the vertices have outdegree $3p + 1$. Without loss of generality, the central vertex v (which is a neighbor of all other vertices) has outdegree $3p + 1$. Hence two consecutive non-central vertices x, y have outdegree p. As there are $2p$ edges between x and y, there is no outgoing edge from $\{x, y\}$. Hence the orientation is not strongly connected, and hence the flow index is precisely $2+\frac{1}{p}$ by Theorem 1.1.

In the examples above we calculate flow indices close to 2, that is, we determine how close an orientation can be to a balanced orientation. It is perhaps interesting to note that the orientations we use in the proof are very far from being balanced, in fact, they are not even strongly connected.

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