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Parity Subgraphs with Few Common Edges and Nowhere-Zero 5-Flow

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Abstract A parity subgraph of a graph is a spanning subgraph such that the degrees of all vertices have the same parity in both the subgraph and the original graph. Let G be a cyclically 6-edge-connected cubic graph. Steffen (Intersecting 1-factors and nowhere-zero 5-flows 1306.5645, 2013) proved that G has a nowhere-zero 5-flow if G has two perfect matchings with at most two intersections. In this paper, we show that G has a nowhere-zero 5-flow if G has two parity subgraphs with at most two common edges, which generalizes Steffen's result.

1 Introduction

Let G be a graph. The set of neighbors of a vertex v in G is denoted by $N_G(v)$. The degree $d_G(v)$ of a vertex v is the number of edges incident with v. An r-regular graph is a graph with each vertex having degree r. A circuit in G is a 2-regular connected graph. An r-factor of G is a spanning r-regular subgraph of G. A perfect matching of G is a 1-factor of G. A subgraph P of G is a parity subgraph if $d_P(v) \equiv d_G(v) \pmod 2$ for all $v \in V(G)$. An even subgraph H of G is a spanning subgraph with the property that $d_H(v) \equiv 0 \mod 2$ for each $v \in V(G)$. So a subgraph P is a parity subgraph of G if and only if $G - P$ is an even subgraph of G. The oddness of G, denoted by $o(G)$, is the minimum number of odd components in a spanning even subgraph of G. For $S \subseteq V(G)$, let $\partial_G(S)$ be the set of edges with one end in S and

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another in $V(G)\backslash S$. If S is a proper subset of $V(G)$, then $\partial_G(S)$ is an edge-cut of G. A bridge is an edge-cut of order one. A cyclic edge-cut F of G is an edge-cut of G such that two components of $G - F$ contain circuits. The cyclic edge-connectivity of G, denoted by n^*_G, is the minimum cardinality of a cyclic edge-cut of G. We call G is cyclically k-edge-connected if $n^*_G = k$.

Let k be a positive integer and let $D(G)$ be an orientation graph of a graph G. Let φ be a function from the set of directed edges of $D(G)$ into the set $\{0, 1, \ldots, k-1\}$. For $S \subseteq V(G)$ let

$$\delta_\varphi(S) = \sum_{e \in \partial^+(S)} \varphi(e) - \sum_{e \in \partial^-(S)} \varphi(e).$$

The function φ is a *k-flow* on G if $\delta_\varphi(S) = 0$ for every $S \subseteq V(G)$. The support of φ is the set $\{e \in E(G) \mid \varphi(e) \neq 0\}$, and it is denoted by $supp(\varphi)$. A k-flow φ is an integer nowhere-zero k-flow (NWZ k-flow for short) on G if $supp(\varphi) = E(G)$. (For more flexible definitions, see e.g. [11]).

In 1954, Tutte proposed the following conjecture.

Conjecture 1.1 (5-Flow conjecture) *Every bridgeless graph has a nowhere-zero* 5-*flow.*

This is one of the famous Tutte's integer flow conjectures still open.

Tutte also proposed a weaker conjecture that there exists an integer $k \geq 5$ such that every bridgeless graph has a NWZ k-flow. This was proved by Jaeger [5,6] (also independently by Kilpatrick [8]) for $k = 8$ (known as 8-flow theorem). This result was improved by Seymour [10] to $k = 6$ (known as 6-flow theorem).

Kochol [9] proved that a minimum counterexample to the 5-flow conjecture is a cyclically 6-edge-connected cubic graph. Hence it suffices to prove 5-Flow Conjecture for these graphs. Jaeger [7] proved that every bridgeless cubic graph with a 2-factor having 0 or 2 odd components has a NWZ 5-flow. Steffen [13] improved Jaeger's result as follows.

Theorem 1.2 (Theorem 1.2 in [13]) *Every cyclically 6-edge-connected cubic graph with two perfect matchings having at most two intersections has a NWZ 5-flow.*

Clearly, for cubic graphs, a perfect matching is a parity subgraph, but not vice versa. For example, Fig. 1(2) and (3) give two parity subgraphs of Petersen graph with one common edge and two common edges, respectively.

In this paper, we generalize Steffen's result to parity subgraphs with few common edges. The main result is the following.

Theorem 1.3 *Let G be a cyclically 6-edge-connected cubic graph and let P_1, P_2 be two parity subgraphs of G. If $|P_1 \cap P_2| \leq 2$, then G has a NWZ 5-flow.*

2 Preliminaries

Lemma 2.1 *If G is a bridgeless cubic graph and P_1, P_2 are two parity subgraphs of G, then $o(G) \leq 2|P_1 \cap P_2|$.*

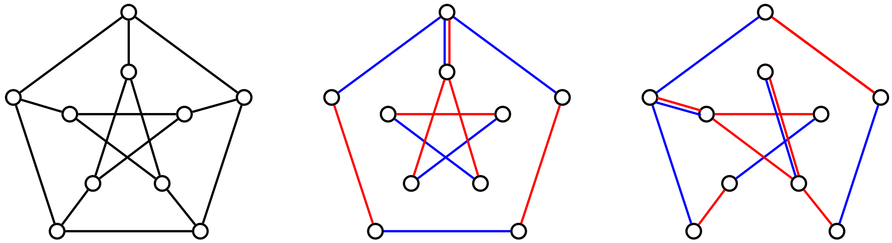


Fig. 1 Petersen graph and its parity subgraphs, where the two parity subgraphs are colored by *red* and *blue*, respectively, and the intersections are *bicolored*

Proof Let $\overline{P_1} = G - P_1$. Then $\overline{P_1}$ is an even subgraph. Let C be an odd component of $\overline{P_1}$. Then $\partial_G(V(C)) \subseteq P_1$. Since P_2 is a parity subgraph of G, $|\partial_G(V(C)) \cap P_2| \equiv |\partial_G(V(C))| \mod 2$. That is $\partial_G(V(C)) \cap P_2 \neq \emptyset$. Since one edge in $P_1 \cap P_2$ appears in at most two of $\partial_G(V(C))$ and C is an odd component of $\overline{P_1}$, $o(G) \leq 2|P_1 \cap P_2|$. □

Let A be an Abelian group with additive notation. A nowhere-zero A-flow (NWZ A-flow for short) on G is an assignment of a direction and a value of $A \backslash \{0\}$ to each edge of G such that the sum of the values of outgoing edges is equal to the sum of the values of ingoing edges at every vertex of G.

A graph G is called an F_k-graph (for $k \geq 2$) (defined in [6,7]) if it satisfies the following equivalent properties:

(a) for some additive group A of order k, G has a nowhere-zero A-flow;
(b) for every additive group A of order k, G has a nowhere-zero A-flow;
(c) G has a nowhere-zero k-flow.

A graph G is called a nearly F_4-graph if it is possible to add a new edge to G in order to obtain an F_4-graph (defined in [7]).

The following two lemmas can be found in [7] or [3].

Lemma 2.2 (Theorem 3.6 in [7], or Theorem 4.5 in [3]) *A bridgeless cubic graph G admits a NWZ 4-flow if and only if G is 3-edge-colorable.*

Lemma 2.3 (Theorem 8.1 in [7]) *Every bridgeless nearly F_4-graph is an F_5-graph.*

Jaeger [7] proved that bridgeless cubic graph with a 2-factor having 0 or 2 odd components is nearly F_4-graph and so is an F_5-graph. The following lemma is an analogous result.

Lemma 2.4 *If G is a bridgeless cubic graph with an even subgraph having 0 or* 2 *odd components, then G has a NWZ* 5*-flow.*

Proof By Lemma 2.3, it suffices to show that G is a nearly F_4-graph. If the even subgraph has no odd components, then G is 3-edge colorable and hence a F_4-graph by Lemma 2.2. If the even subgraph has two odd components, say C and C' (C and C' may be isolated vertices), then its edges can be colored with colors (0, 1) and (1, 0) in such a way that each vertex, with exceptions of two vertices $v \in V(C)$ and $v' \in V(C')$, is bicolored; if we now join v and v' by a new edge and color all the uncolored edges by (1, 1), then the resulting coloring of edges defines a NWZ Z_2^2-flow. □

An orientation D of G is an assignment of a direction to each edge. For $S \subseteq V(G)$, let $\partial_G^+(S)$ (resp. $\partial_G^-(S)$) be the set of outgoing (resp. ingoing) edges of $\partial_G(S)$. The oriented graph is denoted by $D(G)$, $d_{D(G)}^-(v) = |\partial_G^-(\{v\})|$ and $d_{D(G)}^+(v) = |\partial_G^+(\{v\})|$ denote the indegree and outdegree of vertex v in $D(G)$, respectively.

We will use balanced valuations of graphs, which were introduced by Bondy [1] and Jaeger [4]. A balanced valuation of a graph G is a function f from the vertex set $V(G)$ into the real numbers, such that for all $X \subseteq V(G) : |\sum_{v\in X} f(v)| \le |\partial_G(X)|$. The following fundamental theorem is given by Jaeger.

Theorem 2.5 (Jaeger [4]) *Let G be a graph with orientation D and $k \ge 3$. Then G has a NWZ k-flow if and only if there is a balanced valuation f of G with $f(v) = \frac{k}{k-2}\left(2d_{D(G)}^+(v) - d_G(v)\right)$, for all $v \in V(G)$.*

In particular, Theorem 2.5 says that a cubic graph G has a NWZ 4-flow (resp. NWZ 5-flow) if and only if there is a balanced valuation of G with values in $\{\pm 2\}$ (resp. $\{\pm\frac{5}{3}\}$).

If we describe a flow which relies on a specific orientation D of the edges of G, then we also write (D, φ). For $i \in \{1, 2\}$, let (D_i, φ_i) be flows on G. The sum $(D_1, \varphi_1) + (D_2, \varphi_2)$ is the flow (D, φ) on G with orientation

$$D = D_1|_{\{e:\ \varphi_1(e)\ge\varphi_2(e)\}} \cup D_2|_{\{e:\ \varphi_2(e)>\varphi_1(e)\}},$$

and with flow value

$$\varphi(e) = \begin{cases} \varphi_1(e) + \varphi_2(e), & \text{if } e \text{ received the same direction in } D_1 \text{ and } D_2; \\ |\varphi_1(e) - \varphi_2(e)|, & \text{otherwise.} \end{cases}$$

Let G be a cubic 3-edge-colorable graph and let c be a 3-edge-coloring of G. A *canonical NWZ 4-flow* of G with respect to c is defined as follows (Steffen [13]): For $i, j \in \{1, 2, 3\}$ with $1 \le i < j \le 3$, let $H_{i,j}$ be the 2-factor of G induced by the edges $c^{-1}(i) \cup c^{-1}(j)$. Let $\varphi_{1,2}$ be the flow on the directed circuits of $H_{1,2}$ with $\varphi_{1,2}(e) = 1$ for all $e \in E(H_{1,2})$, and let $\varphi_{2,3}$ be the flow on the directed circuits of $H_{2,3}$ with $\varphi_{2,3}(e) = 2$ for all $e \in E(H_{2,3})$. Then $\varphi = \varphi_{1,2} + \varphi_{2,3}$ is a NWZ 4-flow on G. Note that the edges of $c^{-1}(1)$ have flow value 1, the edges of $c^{-1}(2)$ have flow value 1 or 3, and the edges of $c^{-1}(3)$ have flow value 2. The circuits of $H_{2,3}$ are directed circuits in $D(G)$.

By the construction of the canonical NWZ 4-flow φ, φ induces a balanced valuation f of G with $f(v) = 2(2d_{D(G)}^+(v) - d_G(v)) \in \{\pm 2\}$ for all $v \in V$. Let $A = \{v \mid f(v) = -2\}$ and $B = \{v \mid f(v) = 2\}$. Then A and B forms a partition of $V(G)$. A balanced valuation which is induced by a canonical NWZ 4-flow is called a canonical balanced valuation on G.

By the construction of canonical NWZ 4-flow, we have the following observation.

Observation 2.6 *Let c be a 3-edge-coloring of a cubic graph G, and let A, B be a partition of $V(G)$ which is induced by a canonical NWZ 4-flow with respect to c. If $e = xy \in c^{-1}(1) \cup c^{-1}(2)$, then the following holds.*

(i) ([13]) The two ends of e belong to different classes, i.e. $x \in A$ if and only if $y \in B$.
(ii) $e \in c^{-1}(1)$ and $x \in A$ if and only if e is orientated from y to x.

Let $G[S]$ be the subgraph induced by the set S of vertices in a graph G.

Lemma 2.7 *Let G be a cubic graph and $T \subseteq V(G)$. If $|T| > |\partial_G(T)| - 2$, then $G[T]$ contains circuit.*

Proof Since $|E(G[T])| = \frac{1}{2}(3|T| - |\partial_G(T)|) = |T| - 1 + \frac{1}{2}(|T| + 2 - |\partial_G(T)|)$ and $|T| > |\partial_G(T)| - 2$, $|E(G[T])| > |T| - 1$. Hence $G[T]$ is not a tree. □

Lemma 2.8 (Menger's theorem (directed vertex-disjoint version)) *Let $D = (V, A)$ be a digraph and let $S, T \subseteq V$. Then the maximum number of vertex-disjoint directed paths from S to T is equal to the minimum size of vertex set separating S from T.*

3 Proof of Theorem 1.3

The first step, we claim that if $|P_1 \cap P_2| \leq 1$, then G admits a NWZ 5-flow.

Claim 1 *If $|P_1 \cap P_2| \leq 1$, then G has a NWZ 5-flow.*

Proof By Lemma 2.1, $o(G) \leq 2|P_1 \cap P_2| \leq 2$. The result follows directly from Lemma 2.4. □

Now, it suffices to show that G has a NWZ 5-flow if G is cyclically 6-edge-connected and $|P_1 \cap P_2| = 2$.

Again by Lemma 2.4, we have

Claim 2 *If $|P_1 \cap P_2| = 2$ and $o(G) = 2$, then G has a NWZ 5-flow.*

Now we consider the case that $|P_1 \cap P_2| = 2$ and $o(G) = 4$. Let $P_1 \cap P_2 = \{e_1, e_2\}$ and let $e_i = v_i w_i$ for $i = 1, 2$.

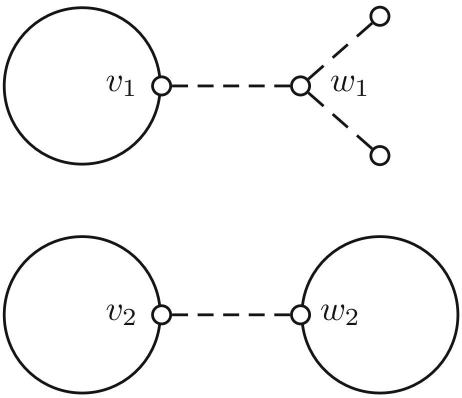
Recall that P_1 and P_2 are parity subgraphs of the cubic graph G. For each vertex $v \in V(G)$, $d_{P_i}(v) = 1$ or 3, i=1, 2 and $P_1 \triangle P_2$ is an even subgraph of G, where $P_1 \triangle P_2$ denote the symmetric difference of P_1 and P_2, i.e. the subgraph of G induced by the edge set $[E(P_1) \cup E(P_2)] - [E(P_1) \cap E(P_2)]$.

Claim 3 *G has an even subgraph H having precisely four odd components with the property that at most one component is an isolated vertex, and each component contains precisely one vertex of $\{v_1, v_2, w_1, w_2\}$ (see Fig. 2).*

Proof If one of P_1, P_2, say P_1, has at most one vertex of degree three, then $H = G - P_1$ is an even subgraph of G as desired.

Otherwise, both P_1 and P_2 have precisely two vertices of degree 3, then $P = P_1 \triangle P_2$, is a 2-factor of G with precisely four odd circuits, as desired. □

Fig. 2 The even subgraph H



Let C_x be the odd component of H containing x for $x \in \{v_1, v_2, w_2, w_1\}$, where C_{w_1} is either an odd circuit or an isolated vertex.

Let G' be the suppressed graph obtained from $G - \{e_1, e_2\}$ by suppressing the four degree 2 vertices v_1, v_2, w_1, w_2, denote the new edge by suppressing x by e_x in G' for $x \in \{v_1, v_2, w_1, w_2\}$. Then H', the subgraph of G' corresponding to H, is a 2-factor of G' having no odd component and so G' is a 3-edge-colorable cubic graph.

Let c' be a 3-edge-coloring of G' such that the edges of H' are colored with colors 2 and 3 such that e_{v_1}, e_{v_2} and e_{w_2} are colored with color 3, and e_{w_1} are colored with color 3 if e_{w_1} is an edge in a circuit of H', with color 1 if $e_{w_1} \notin H'$. Let φ' be the canonical NWZ 4-flow with respect to c', and let A' and B' be the partition of $V(G')$ with respect to the canonical balanced valuation, say f', on G' induced by φ'. Note that subdividing an edge does not change flow properties of graphs. Hence φ' induces a NWZ 4-flow φ'' on $G - \{e_1, e_2\}$. Set $A = A' \cup \{v_1, v_2\}$ and $B = B' \cup \{w_1, w_2\}$. Then A and B is a partition of $V(G)$. Define a map f from $V(G)$ to $\{\pm\frac{5}{3}\}$ with $f(v) = \frac{5}{3}$ if $v \in A$ and $f(v) = -\frac{5}{3}$ if $v \in B$.

In the following we claim that f is a balanced valuation on G and hence G has a NWZ 5-flow by Theorem 2.5.

Suppose to the contrary that G does not have a NWZ 5-flow. Then f is not a balanced valuation. Hence there is a subset $S \subseteq V(G)$ such that $|\sum_{v \in S} f(v)| > |\partial_G(S)|$. Let $k = ||S \cap A| - |S \cap B||$. Then, $\frac{5}{3}k > |\partial_G(S)|$.

Claim 4 $|\partial_G(S)| = 6$ *and* $k = 4$.

Proof Note that $|\partial_G(S)| = 3|S| - 2|E(G[S])| = 3(|S \cap A| + |S \cap B|) - 2|E(G[S])|$ and $k = ||S \cap A| - |S \cap B||$. We have

$$k \equiv |\partial_G(S)| \pmod 2. \tag{1}$$

Moreover, for $X \in \{S, \overline{S}\}$, if $G[X]$ contains no circuit, then $|E(G[X])| \leq |X| - 1$. Hence

$$|\partial_G(S)| = |\partial_G(X)| \geq 3|X| - 2(|X| - 1) = |X| + 2 \geq 3. \tag{2}$$

Otherwise, $|\partial_G(S)| \geq 6$, since G is cyclically 6-edge-connected.

Let $S' = (S \cap A') \cup (S \cap B')$ and let $k' = ||S \cap A'| - |S \cap B'||$. Then $|\partial_{G'}(S')| \geq |\sum_{v \in S'} f'(v)| = 2k'$, since f' is a canonical balanced valuation on G' induced by the NWZ 4-flow φ' of G'.

If $|\{e_1, e_2\} \cap \partial_G(S)| = 0$, then either both of the two ends of e_i $(i = 1, 2)$ belong to S or neither of them belongs to S. Hence $k' = k$. So, $2k \leq |\partial_{G'}(S')| = |\partial_G(S)| < \frac{5}{3}k$, a contradiction.

If $|\{e_1, e_2\} \cap \partial_G(S)| = 1$, without loss of generality, assume $\{e_1, e_2\} \cap \partial_G(S) = \{e_1\}$ and $v_1 \in S$, then $k' = k - 1$ or $k + 1$. In the worst case, $2(k - 1) \leq |\partial_{G'}(S')| = |\partial_G(S)| - 1 < \frac{5}{3}k - 1$, that is $k < 3$. By (2), $3 \leq |\partial_G(S)| < \frac{5}{3}k$. This implies that $k = 2$ and $|\partial_G(S)| = 3$, a contradiction with Eq. (1).

If $|\{e_1, e_2\} \cap \partial_G(S)| = 2$, then each of e_i $(i = 1, 2)$ has one end in S and the another one in $\overline{S}$. Hence, $k - 2 \leq k' \leq k + 2$. So, in the worst case, $2(k - 2) \leq |\partial_{G'}(S')| = |\partial_G(S)| - 2 < \frac{5}{3}k - 2$, that is $k < 6$. By (2), $3 \leq |\partial_G(S)| < \frac{5}{3}k$. Hence the possible choices for $(k, |\partial_G(S)|)$ are (2, 3), (3, 4), (4, 6) and (5, 8). By Eq. (1), k must be 4 and $|\partial_G(S)| = 6$. □

Let $\partial_G(S) = \{e_1, e_2, f_1, f_2, f_3, f_4\}$ and let $f_i = x_i y_i$ $(i = 1, 2, 3, 4)$. By Claim 4, we may assume $\{v_1, v_2, x_1, x_2, x_3, x_4\} \subseteq S$. Let φ be the 4-flow on G with $supp(\varphi) = E(G) - \{e_1, e_2\}$, which is obtained from the (canonical) NWZ 4-flow φ'' on $G - \{e_1, e_2\}$. Let $G'' = G - \{e_1, e_2\}$.

Without loss of generality, assume $k = |S \cap A| - |S \cap B|$(or we can choose the orientation with an opposite direction). Let S' and k' be the same meaning as in the proof of Claim 4. Then $k' = |S' \cap A'| - |S' \cap B'| = 2$ and $\partial_{G'}(S') = \{f_1, f_2, f_3, f_4\}$.

Note that, for each circuit C of the 2-factor $c'^{-1}(1) \cup c'^{-1}(2)$, $|E(C) \cap \partial_{G'}(S')| \equiv 0 \mod 2$ and $|V(C) \cap A'| = |V(C) \cap B'|$ by Lemma 2.6 (i). So, there are at most two circuits C_1 and C_2 (maybe $C_1 = C_2$) of the 2-factor $c'^{-1}(1) \cup c'^{-1}(2)$ with the property that $|E(C_i) \cap \partial_{G'}(S')| = 2$ for $i = 1, 2$ and $|S' \cap A' \cap (V(C_1) \cup V(C_2))| - |S' \cap B' \cap (V(C_1) \cup V(C_2))| = k' = 2$. That is two of $\{f_1, f_2, f_3, f_4\}$ are colored with color 1, say f_1, f_2, and the others are colored with color 2, say f_3, f_4, and, moreover, $\{x_1, x_2, x_3, x_4\} \in A'$.

By Lemma 2.6 (ii), f_i is oriented from y_i to x_i for $i = 1, 2$. Then $\varphi'(f_1) = \varphi'(f_2) = 1$ by the construction of φ'. Since $\varphi'(f_i) = 1$ or 3 for $i = 3, 4$, to guarantee that $\delta_{\varphi'}(S') = 0$, one of $\{f_3, f_4\}$, say f_3, is oriented from y_3 to x_3 with $\varphi'(f_3) = 1$, the other one, say f_4, is oriented from x_4 to y_4 with $\varphi'(f_4) = 3$.

Let

$$L_i = \{z \in S \mid \quad \text{There is a directed path from} \quad v_i \quad \text{to} \quad z \quad \text{in} \quad D(G[S])\}, i = 1, 2$$

and

$$T_2 = \{z \in \overline{S} \mid \quad \text{There is a directed path from} \quad z \quad \text{to} \quad w_2 \quad \text{in} \quad D(G[\overline{S}])\}.$$

Then $L_i \neq \emptyset$ since $v_i \in L_i$ for $i = 1, 2$, and $T_2 \neq \emptyset$ since $w_2 \in T_2$.

Claim 5 *If* $|L_i| \neq 2$, *then* $v_{3-i} \in L_i$, *that is there is a directed path from* v_i *to* v_{3-i} *in* $D(G[S])$ *for* $i = 1, 2$.

Proof We prove the case $i = 2$; the case $i = 1$ can be proved similarly.

Suppose to the contrary that $v_1 \notin L_2$. Since L_2 has no outgoing edges in $D(G[S])$, f_4 is the only possible outgoing edge of L_2 in G'. Hence $\partial^+_{G'}(L_2) = \emptyset$ or $\{f_4\}$. Since G is bridgeless, $\partial^+_{G'}(L_2) \neq \emptyset$ and so $\partial^+_{G'}(L_2) = \{f_4\}$.

Recall that φ'' is a NWZ 4-flow of $G''(= G - \{e_1, e_2\})$ induced by the canonical NWZ 4-flow φ' on G'. Hence $\delta_{\varphi''}(L_2) = 0$. Since $\varphi''(f_4) = 3$, $\partial^-_{G''}(L_2) \leq 3$. It follows that

$$|\partial_G(L_2)| = |\partial^+_{G'}(L_2) \cup \partial^-_{G''}(L_2) \cup \{e_2\}| = |\partial^-_{G''}(L_2)| + 2 \leq 5.$$

Since $|L_2| \neq 2$, $|L_2| \geq 3$.

If $|\partial^-_{G''}(L_2)| \leq 2$, then $|L_2| > |\partial_G(L_2)| - 2$. Hence $G[L_2]$ contains circuits by Lemma 2.7.

If $|\partial^-_{G''}(L_2)| = 3$, then each edge of $\partial^-_{G''}(L_2)$ has flow value 1 since the only possible outgoing edge f_4 has flow value 3. We claim that $|L_2| \geq 4$. Suppose to the contrary that $|L_2| = 3$. Let $L_2 = \{x_4, v_2, u\}$. Then $|E(G[L_2])| = \frac{1}{2}(3|L_2| - |\partial_G(L_2)|) = 2$ since $\partial_G(L_2)$ has five edges. Hence, by the definition of L_2, $D(G[L_2])$ is a directed path starting at v_2 and ending at x_4 or u. This implies that there is an ingoing edge of $\partial_{G''}(L_2)$ incident with v_2. But this ingoing edge has flow value 2 by the construction of the NWZ 4-flow φ'', a contradiction. Therefore $|L_2| \geq 4$. Again by Lemma 2.7, $G[L_2]$ contains circuits.

Note that the circuit C_x is a directed circuit in G'' for $x \in \{v_1, v_2, w_2\}$. Then at most one of $C_{v_1}, C_{v_2}, C_{w_2}$ intersects with $\partial_G(S)$. Since $v_1 \notin L_2$ and $w_2 \notin L_2$, at least one of the circuits C_{v_1}, C_{w_2} is contained in $G[\overline{L_2}]$. It follows that $\partial_G(L_2)$ is a cyclic edge-cut with $|\partial_G(L_2)| \leq 5$, a contradiction with $n^*_G = 6$. □

With a similar discussion, we have the following analogous claim.

Claim 6 *If $|T_2| \neq 2$, then $w_1 \in T_2$, that is there is a directed path from w_1 to w_2 in $D(G[\overline{S}])$.*

If $|L_2| \neq 2$ and $|T_2| \neq 2$, then, by Claim 5, there is a directed path P_1 from v_2 to v_1 in $D(G[S])$, and, by Claim 6, there is a directed path P_2 from w_1 to w_2 in $D(G[\overline{S}])$. Orient the edges e_1 and e_2 appropriately such that the circuit C with $E(C) = E(P_1) \cup E(P_2) \cup \{e_1, e_2\}$ is a directed circuit. Let φ_2 be a 2-flow on G with $\varphi_2(e) = 1$, if $e \in E(C)$, and $\varphi_2(e) = 0$, otherwise. Then $\varphi + \varphi_2$ is a NWZ 5-flow on G, a contradiction with our assumption that G has no NWZ 5-flow.

Now, assume $|L_2| = 2$ or $|T_2| = 2$.

Claim 7 *(1) If $|T_2| = 2$, then $f_4 \in E(C_{w_2})$, $T_2 = \{y_4, w_2\}$ and there is a directed path from y_4 to w_1 in $D(G[\overline{S}])$.*
(2) If $|L_2| = 2$, then $f_4 \in E(C_{v_2})$ and $L_2 = \{x_4, v_2\}$.

Proof (1) Since f_4 is the only ingoing edge of T_2 in G'', then $y_4 \in T_2$ and $T_2 = \{w_2, y_4\}$. If $f_4 \notin E(C_{w_2})$, then $V(C_{w_2}) \in T_2$ (note that C_{w_2} is a directed circuit in G''), a contradiction with $|T_2| = 2$. Hence $f_4 \in E(C_{w_2})$.

Let

$$M_2 = \{z \in \overline{S} \mid \quad \text{there is a directed path from} \quad z \quad \text{to} \quad w_1 \quad \text{in } D\,(G[\overline{S}])\}.$$

If $y_4 \notin M_2$, then $\partial^-_{G''}(M_2) = \emptyset$ since f_4 is the only possible ingoing edge of M_2 in G''. Hence $\partial^+_{G''}(M_2) = \emptyset$ and so $\partial_G(M_2) = \{e_1\}$, a contradiction with G being bridgeless. Hence $y_4 \in M_2$, i.e. there is a directed path from y_4 to w_1 in $D(G[\overline{S}])$.

(2) It can be proved similarly to the statement (1). □

If $|T_2| = 2$, by Claim 7 (1), $f_4 \in E(C_{w_2})$, $T_2 = \{y_4, w_2\}$ and there is a directed path, say $P(y_4, w_1)$, from y_4 to w_1 in $D(G[\overline{S}])$. Since $f_4 \in E(C_{w_2})$, $f_4 \notin E(C_{v_1})$ and $f_4 \notin E(C_{v_2})$. Hence $V(C_{v_i}) \subseteq S$ for $i = 1, 2$. So, $|L_i| \geq 3, i = 1, 2$. By Claim 5, there is a directed path, say $P(v_1, v_2)$ from v_1 to v_2 in $D(G[S])$. Orient the edges e_1 and e_2 appropriately such that the circuit C with $E(C) = E(P(v_1, v_2)) \cup E(P(y_4, w_1)) \cup \{e_1, e_2\} \cup \{y_4 w_2\}$ is a nearly directed circuit (the edge $y_4 w_2$ is the only possible edge having a reverse direction and $\varphi''(y_4 w_2) = 2$, $y_4 w_2$ maybe not in C if $w_2 \in V(P(y_4, w_1))$). Let φ_2 be a 2-flow on G with $\varphi_2(e) = 1$, if $e \in E(C)$, and $\varphi_2(e) = 0$, otherwise. Then $\varphi + \varphi_2$ is a NWZ 5-flow on G (note that $(\varphi + \varphi_2)(y_4 w_2) = 1$ if $y_4 w_2 \in E(C)$), a contradiction with our assumption that G has no NWZ 5-flow.

If $|L_2| = 2$, by Claim 7 (2), $f_4 \in E(C_{v_2})$. Hence $f_4 \notin E(C_{v_1})$ and $f_4 \notin E(C_{w_2})$. So $V(C_{v_1}) \subseteq L_1$ and $V(C_{w_2}) \subseteq T_2$. This implies that $|L_1| \geq 3$ and $|T_2| \geq 3$. By Claim 5, there is a directed path, say $P(v_1, v_2)$, from v_1 to v_2 in $D(G[S])$ and, by Claim 6, there is a directed path, say $P(w_1, w_2)$, from w_1 to w_2 in $D(G[\overline{S}])$.

Let

$$M_1 = \{z \in S \mid \quad \text{there is a directed path from} \quad z \quad \text{to} \quad v_1 \quad \text{in} \quad D(G[S])\}$$

and

$$M_3 = \{z \in \overline{S} \mid \quad \text{there is a directed path from} \quad w_2 \quad \text{to} \quad z \quad \text{in} \quad D(G[\overline{S}])\}.$$

If $v_2 \in M_1$ or $w_1 \in M_3$, then we can get a directed circuit C with $E(C) = E(P(w_1, w_2)) \cup E(P(v_2, v_1)) \cup \{e_1, e_2\}$ or $E(P(v_1, v_2)) \cup E(P(w_2, w_1)) \cup \{e_1, e_2\}$ by orienting e_1 and e_2 appropriately, where $P(v_2, v_1)$ is a directed path from v_2 to v_1 in $D(G[S])$ and $P(w_2, w_1)$ is a directed path from w_2 to w_1 in $D(G[\overline{S}])$. Hence we can define a 2-flow φ_2 on G with $\varphi_2(e) = 1$, if $e \in E(C)$, and $\varphi_2(e) = 0$, otherwise, such that $\varphi + \varphi_2$ is a NWZ 5-flow on G, again a contradiction with our assumption that G has no NWZ 5-flow.

Now assume $v_2 \notin M_1$ and $w_1 \notin M_3$. Since M_1 has no ingoing edges in $D(G[S])$, $\partial^-_{G'}(M_1) \subseteq \{f_1, f_2, f_3\}$.

If $|\partial^-_{G'}(M_1)| \leq 2$, then $|\partial^+_{G'}(M_1)| \leq |\partial^-_{G'}(M_1)| \leq 2$ since $\varphi'(f_i) = 1$ for $i = 1, 2, 3$. Hence $|\partial_G(M_1)| = |\partial_{G'}(M_1) \cup \{e_1\}| \leq 5$. Since $f_4 \notin E(C_{v_1})$ and $f_4 \notin E(C_{w_2})$, $V(C_{v_1}) \subseteq M_1$ and $V(C_{w_2}) \subseteq \overline{M_1}$. Hence $\partial_G(M_1)$ is a cyclic edge-cut of G, a contradiction with $n^*_G = 6$. Therefore, $\partial^-_{G'}(M_1) = \{f_1, f_2, f_3\}$. With a similar discussion, we can show that $\partial^+_{G'}(M_3) = \{f_1, f_2, f_3\}$ too.

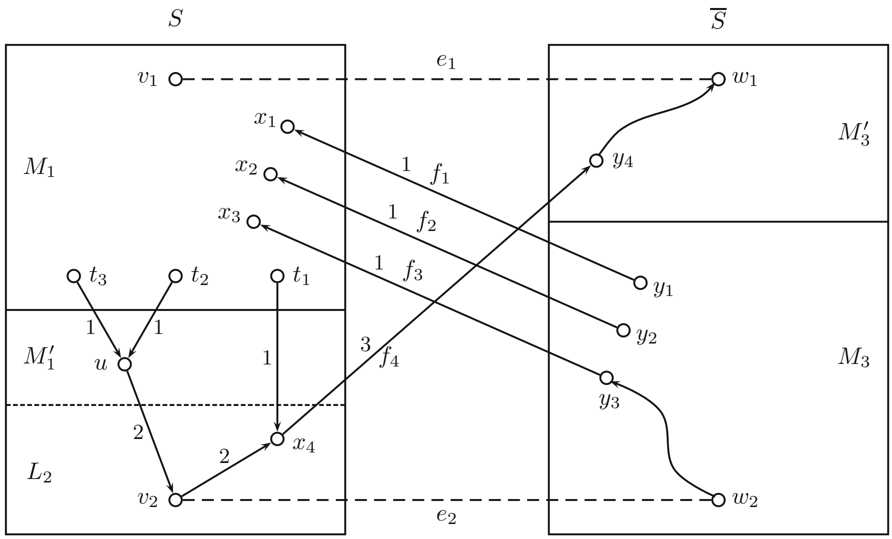


Fig. 3 The graph G'' in Claim 8

Let $M_1' = S - M_1$. Then $L_2 \subseteq M_1'$ and $\partial^+_{G''}(M_1') = \{f_4\}$. Hence $|\partial^-_{G''}(M_1')| \leq 3$ since $\varphi''(f_4) = 3$. Let $M_3' = \overline{S} - M_3$. Since $M_2 \subseteq M_3'$, $y_4 \in M_3'$ and there is a directed path from y_4 to w_1 in $D(G[M_3'])$ by Claim 7.

If $|M_1'| \geq 4$, by Lemma 2.7, $G[M_1']$ contains circuit. But $|\partial_G(M_1')| = |\partial_{G''}(M_1') \cup \{e_2\}| \leq 5$, a contradiction with $n^*_G = 6$.

If $|M_1'| = 2$, then $M_1' = L_2 = \{x_4, v_2\}$. Note that v_2 is a bivalent vertex in G''. Let u be another neighbor of v_2 in G''. Then uv_2 is an ingoing edge of $\partial_{G''}(M_1')$ with $\varphi''(uv_2) = 2$. Hence $|\partial^-_{G''}(M_1')| = 2$ since $\delta_{\varphi''}(M_1') = 0$. Let $\partial^-_{G''}(M_1') = \{t_1x_4, uv_2\}$. Then $\varphi''(t_1x_4) = 1$. Since $u \in M_1$, there is a directed path $P(u, v_1)$ from u to v_1 in $D(G[S])$. Orient the edges e_1, e_2 appropriately, we can get a nearly directed circuit C with $E(C) = E(P(u, v_1)) \cup E(P(w_1, w_2)) \cup \{e_1, e_2\} \cup \{uv_2\}$ (the edge uv_2 has a reverse direction and $\varphi''(uv_2) = 2$). Hence we can define a 2-flow φ_2 on G with $\varphi_2(e) = 1$ if $e \in E(C)$, and $\varphi_2(e) = 0$, otherwise, such that $\varphi + \varphi_2$ is a NWZ 5-flow on G, again a contradiction with our assumption that G has no NWZ 5-flow.

Now assume $|M_1'| = 3$. Let $M_1' = \{x_4, v_2, u\}$. Then M_1' must be a directed path from u to x_4 in G''. Since $\partial^+_{G''}(M_1') = \{f_4\}$ and $\varphi''(f_4) = 3$, $\partial^-_{G''}(M_1')$ contains precisely three ingoing edges with flow value 1. Let $\partial^-_{G''}(M_1') = \{t_1x_4, t_2u, t_3u\}$ and assume $t_3u \in E(C_{v_2})$.

Let P be the strong connected component in $D(G[M_1])$ containing v_1. Then $C_{v_1} \subseteq P$ and $\partial^+_{G''}(P) \subseteq \{t_1x_4, t_2u, t_3u\}$. If $|\partial^+_{G''}(P)| \leq 2$, then $|\partial^-_{G''}(P)| \leq 2$. Hence $\partial_G(P)$ is a cyclic edge-cut of G with at most five edges, a contradiction with $n^*_G = 6$. Hence $\partial^+_{G''}(P) = \{t_1x_4, t_2u, t_3u\}$ with $t_1, t_2, t_3 \in V(P)$ (see Fig. 3).

Claim 8 *(1)* $f_3 \in E(C_{v_2})$.

(2) There are two vertex-disjoint directed paths from $\{v_1, w_2\}$ *to* $\{t_1, t_2, t_3\}$ *in* $D(G'')$. *Moreover, the two vertex-disjoint directed paths have pairwise different ends.*

Proof (1) It follows directly from the structure of $\partial_{G''}(S)$ (C_{v_2} is a directed circuit in G'' with edges of colors 2 and 3, f_3, f_4 are the only two edges with color 2 in $\partial_{G''}(S)$ and $f_4 \in E(C_{v_2})$).

(2) Suppose to the contrary that there is no two such directed paths in $D(G'')$. By Lemma 2.8, there is a vertex y such that each directed path from $\{v_1, w_2\}$ to $\{t_1, t_2, t_3\}$ passes through y. Since P is a strong connected component of $D(G[M_1])$, there is a directed path from v_1 to t_i ($i = 1, 2, 3$) in P. Hence $y \in V(P) \subseteq S$. We claim that $y \notin V(C_{v_1})$. If not, then the directed path composed by the directed path from w_2 to y_3 in $D(G[M_3])$, and the segment of C_{v_2} from y_3 to t_3 in $D(G[S])$ does not pass through y, a contradiction.

Let $Q \subseteq V(P)$ be the set of vertices such that all the directed paths from v_1 to $v \in Q$ do not pass through y. Since y is a vertex of degree 3, the vertex y has only one ingoing edge or only one outgoing edge. If y has only one ingoing edge, let $Q' = Q \setminus \{y\}$, then $|\partial^+_{G''}(Q')| = 1$. If y has only one outgoing edge, let $Q' = Q$, then $|\partial^+_{G''}(Q')| = 1$. Hence, in either case, we have $|\partial^-_{G''}(Q')| \leq 3$. Therefore, $|\partial_G(Q')| = |\partial_{G''}(Q') \cup \{e_1\}| \leq 5$. Since $y \notin V(C_{v_1})$, $V(C_{v_1}) \subseteq Q'$. Since $C_{w_2} \in \overline{S}$, $V(C_{w_2}) \subseteq \overline{Q'}$. Hence $\partial_G(Q')$ is a cyclic edge-cut with $|\partial_G(Q')| \leq 5$, a contradiction with $n^*_G \geq 6$.

The two vertex-disjoint directed paths have pairwise different ends follows directly from the fact that v_1, w_2 have out-degree 1 and each t_i ($i = 1, 2, 3$) has in-degree 1 (since $t_i \in P$) in G''. □

By Claim 8, we have two vertex-disjoint directed paths, one from v_1 to t_i, say $P(v_1, t_i)$, another one from w_2 to t_j ($i \neq j$), say $P(w_2, t_j)$, $i, j \in \{1, 2, 3\}$. By Claim 7, there is a directed path from y_4 to w_1, say $P(y_4, w_1)$ in $D(G[M_3'])$. Let $P(t_i, x_4)$ be the directed path in $D(G[S])$ from t_i to x_4 ($P(t_i, x_4) = t_i x_4$ if $i = 1$, or $t_i u v_2 x_4$ if $i = 2, 3$). Then C_1 with $E(C_1) = E(P(v_1, t_i)) \cup E(P(t_i, x_4)) \cup \{f_4\} \cup E(P(y_4, w_1)) \cup \{e_1\}$ is a directed circuit in G by orienting e_1 from w_1 to v_1. Let φ_1 be a 2-flow on G with $\varphi_1(e) = 1$ if $e \in E(C_1)$, and $\varphi_1(e) = 0$, otherwise.

Let $P(t_j, v_2) = t_j x_4 v_2$ if $j = 1$, or $t_j u v_2$ if $j = 2, 3$. Then C_2 with $E(C_2) = E(P(w_2, t_j)) \cup E(P(t_j, v_2)) \cup \{e_2\}$ is a nearly directed circuit in G by orienting e_2 from v_2 to w_2 ($x_4 v_2$ is the only possible edge with reverse direction on C_2). Let φ_2 be a 2-flow on G with $\varphi_2(e) = 1$ if $e \in E(C_2)$, and $\varphi_2(e) = 0$, otherwise.

Since there is no directed path from w_2 to w_1 in $D(G[\overline{S}])$, $P(y_4, w_1)$ and $P(w_2, t_j)$ are vertex-disjoint in $\overline{S}$. Hence C_1 and C_2 have at most one common edge uv_2. Note that $\varphi(uv_2) = \varphi(v_2 x_4) = 2$. Hence $\varphi + \varphi_1 + \varphi_2$ is a NWZ 5-flow on G, a contradiction with the assumption that G has no NWZ 5-flow. □

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