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ORIGINAL PAPER

A Note on Shortest Sign-Circuit Cover of Signed 3-Edge-Colorable Cubic Graphs

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Abstract

A sign-circuit cover $\mathcal{F}$ of a signed graph (G, σ) is a family of sign-circuits which covers all edges of (G, σ). The shortest sign-circuit cover problem was initiated by Máčajová, Raspaud, Rollová, and Škoviera (JGT 2016) and received many attentions in recent years. In this paper, we show that every flow-admissible signed 3-edge-colorable cubic graph (G, σ) has a sign-circuit cover with length at most $\frac{20}{9}|E(G)|$.

Keywords Signed graph · Sign-circuit cover · Cubic graph

1 Introduction

In this paper, graphs may have parallel edges and loops. A *cycle* is a connected 2-regular graph. A graph is *even* if every vertex has even degree, and an Eulerian graph is a connected even graph. A *cycle cover* $\mathcal{C}$ of an ordinary graph is a family of cycles which covers all edges of G. We call $\mathcal{C}$ a *cycle k-cover* of G if $\mathcal{C}$ covers every edge of G exactly k times. The *length* of a cycle cover $\mathcal{C}$ is defined as $\ell(\mathcal{C}) = \sum_{C \in \mathcal{C}} |E(C)|$.

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Determining the shortest length of a cycle cover of a graph G (denoted by $scc(G) = \min\{\ell(\mathcal{C}) : \mathcal{C}$ is a cycle cover$\}$) is a classic optimization problem initiated by Itai, Lipton, Papadimitriou, and Rodeh [8]. Thomassen [20] showed that it is NP-complete to determine whether a bridgeless graph has a Cycle cover with length at most k for a given integer k. A well-known conjecture, the Shortest Cycle Cover Conjecture, was proposed by Alon and Tarsi [1] as follows.

Conjecture 1.1 (*Shortest Cycle Cover Conjecture*) *For any* 2*-edge-connected graph* G, $scc(G) \le \frac{7}{5}|E(G)|$.

The upper bound is achieved by the Petersen graph. Jamshy and Tarsi [9] proved that Conjecture 1.1 implies the well-known Cycle Double Cover Conjecture proposed by Seymour [18] and Szekeres [19]. The best known general result about Conjecture 1.1 is obtained by Bermond, Jackson, Jaeger [2] and Alon, Tarsi [1], independently.

Theorem 1.2 (*Bermond et al.* [2], *Alon and Tarsi* [1]]) *Let* G *be a* 2*-edge-connected graph. Then* $scc(G) \le \frac{5}{3}|E(G)|$.

Several improvements of this upper bound for cubic graphs G have been made in literature. Specifically, Jackson [12] showed that $scc(G) \le \frac{64}{39}|E(G)|$ and Fan [6] later showed that $scc(G) \le \frac{44}{27}|E(G)|$, Kaiser et al. [11] improved Fan's result to $scc(G) \le \frac{34}{21}|E(G)|$ and Hou and Zhang [7] proved that $scc(G) \le \frac{8}{5}|E(G)|$ if G has girth at least 7 and $scc(G) \le \frac{361}{225}|E(G)|$ if all 5-cycles of G are disjoint. Recently, Lukoťka [15] showed that $scc(G) \le \frac{212}{135}|E(G)|$ for all 2-edge-connected cubic graphs G. For more results about cycle cover of graphs, one can refer [17, 24].

A *signed* graph (G, σ) is a graph G associated with a mapping $\sigma : E(G) \to \{+1, -1\}$. An edge $e \in E(G)$ is *positive* if $\sigma(e) = 1$ and *negative* if $\sigma(e) = -1$. A signed graph G is called *positive* if G contains an even number of negative edges and otherwise called *negative*. In a signed graph, a cycle with an even number of negative edges is called a *balanced cycle*, and otherwise we call it an *unbalanced cycle*. A *barbell* is a signed graph consisting of two unbalanced cycles joined by a (possibly trivial) path, intersecting with the cycles only at ends. If the path in a barbell is trivial, the barbell is called a *short barbell*; otherwise, it is a *long barbell*.

A balanced cycle or a barbell is called a *sign-circuit* of a signed graph.

Remark We should mention that a circuit in a graph or a signed graph, in the matroid sense, can be defined as a minimal dependent set of graphical matroid or signed-graphical matroid. It has been proved that circuits in graphical matroid or signed-graphical matroid are equivalent to the cycles and sign-circuits of graphs or signed graphs (see [23]). To avoid the confusion for the readers not familiar with the matroid theory, we use cycle in graphs and sign-circuit in signed graphs in this article. A *sign-circuit cover* $\mathcal{F}$ of a signed graph is a family of sign-circuits which covers all edges of (G, σ). In fact, it is well-known that a signed graph has a sign-circuit cover if and only if each edge lies in a sign-circuit, which is equivalent to the fact that the signed graph admits a nowhere-zero integer flow, so-called, *flow-admissible*. (Readers may refer to [3] for details). A collection $\mathcal{C}$ of subgraphs of a signed graph H is a *weak sign-circuit cover* of H if all the elements of $\mathcal{C}$ are sign-circuits and each non-bridge edge of H is covered by $\mathcal{C}$. The shortest length of a sign-circuit cover of a signed graph (G, σ) is also denoted by $scc(G)$. The shortest sign-circuit cover problem was initiated by Máčajová et al. [16] and received many attentions in recent years. It is a major open problem for the optimal upper bound of shortest sign-circuit cover in signed graphs.

Problem 1.3 What is the optimal constant c such that $scc(G) \leq c \cdot |E(G)|$ for every flow-admissible signed graph (G, σ) ?

As remarked in [16], the signed Petersen graph (P, σ) whose negative edges induce a cycle of length five has $scc(P) = \frac{5}{3}|E(P)|$, which indicates $c \geq \frac{5}{3}$. We list some of known results related to Problem 1.3.

(1) $c \leq 11$ by Máčajová et al. [16].
(2) $c \leq \frac{14}{3}$ by Lu et al. [14].
(3) $c \leq \frac{25}{6}$ by Chen and Fan [5].
(4) $c \leq \frac{11}{3}$ by Kaiser et al. [10].
(5) $c \leq \frac{19}{6}$ by Wu and Ye [21].

For any flow-admissible 2-edge-connected cubic signed graph (G, σ), Wu and Ye [22] obtained a better upper bound that $scc(G) \leq \frac{26}{9}|E(G)|$. For ordinary 3-edge-colorable cubic graph G, Bermond et al [2] showed that $scc(G) \leq \frac{4}{3}|E(G)|$. In this article, we focus on the shortest sign-circuit cover of signed 3-edge-colorable cubic graphs and prove the following theorem.

Theorem 1.4 *Every flow-admissible signed* 3*-edge-colorable cubic graph* (G, σ) *has a sign-circuit cover with length at most* $\frac{20}{9}|E(G)|$.

An equivalent version of the Four-Color Theorem states that every 2-edge-connected cubic planar graph is 3-edge colorable. So we have the following corollary.

Corollary 1.5 *Every flow-admissible* 2*-edge-connected cubic planar signed graph* (G, σ) *has a sign-circuit cover with length at most* $\frac{20}{9}|E(G)|$.

Now we introduce more notation and terminologies used in the following sections. Let G be a graph and $T \subseteq V(G)$ with $|T| \equiv 0 (\text{mod } 2)$. A *T-join J* of G with respect to T is a subset of edges of G such that $d_J(v) \equiv 1 (\text{mod } 2)$ if and only if $v \in T$, where $d_J(v)$ denotes the degree of v in the edge-induced subgraph $G[J]$. A T-join is minimum if it has minimum number of edges among all T-joins. Let G' be the graph obtained from a graph G by deleting all the bridges of G. Then the components of G' are called the *bridgeless-blocks* of G. By the definition, a bridgeless-block is either a single vertex or a maximal 2-edge-connected subgraph of G. For a vertex subset $U \subseteq V(G)$, $\delta_G(U)$ denotes the set of edges with one end in U and the other in $V(G) \setminus U$. Let u, v be two vertices in $V(G)$. A (u, v)-path is a path connecting u and v. Let $C = v_1 \ldots v_r v_1$ be a cycle where $v_1, v_2, \ldots, v_r$ appear in clockwise on C. A *segment* of C is the path $v_i v_{i+1} \ldots v_{j-1} v_j$ (where the sum of the index is under modulo r) contained in C and is denoted by $v_i C v_j$. A connected graph

H is called a *cycle-tree* [22] if it has no vertices of degree 1 and all cycles of H are edge-disjoint. In a signed graph, *switching* a vertex u means reversing the signs of all edges incident with u. Two signed graphs are *equivalent* if one can be obtained from the other by a sequence of switching operations, and a signed graph is *balanced* if and only if it is equivalent to an ordinary graph. The set of negative edges of (G, σ) is denoted by $E_N(G, \sigma)$.

The rest of the article is organized as follows. Some basic lemmas about signed graph and T-join and a crucial lemma which deals with a special case in our proof are given in Sect. 2. Then we are able to complete the proof of Theorem 1.4 in Sect. 3 and we will finish with some discussions and remark.

2 Some Lemmas

The following lemma due to Bouchet [3] characterized connected flow-admissible signed graphs.

Lemma 2.1 (*Bouchet* [3]) *A connected signed graph* (G, σ) *is flow-admissible if and only if it is not equivalent to a signed graph with exactly one negative edge and it has no bridge* e *such that* $(G - e, \sigma|_{G-e})$ *has a balanced component.*

Lemma 2.2 (*Li et al.* [13]) *Let* T *be a spanning tree of a signed graph* G*. For every* $e \notin E(T)$*, let* C_e *be the unique cycle contained in* $T + e$*. If the cycle* C_e *is balanced for every* $e \notin E(T)$*, then* G *is balanced.*

Wu and Ye [21] gave a lemma to control the size of a T-join.

Lemma 2.3 (*Wu and Ye* [21]) *Let* G *be a 2-edge-connected graph and* T *be subset of vertices with* $|T|$ *even. Then* G *has a* T*-join of size at most* $\frac{1}{2}|E(G)|$.

The following two results gave upper bounds of $scc(G)$ with G under some constrains.

Lemma 2.4 (*Chen and Fan* [4] *and Kaiser et al.* [10]) *Let* (G, σ) *be a signed graph and suppose that each bridgeless-block of* G *is Eulerian.*

(a) (Corollary 1.5 in [4]) If (G, σ) is flow-admissible, then $scc(G) \leq \frac{3}{2}|E(G)|$.
(b) (Corollary 2.6 in [10]) If the union of all the bridgeless-block of G, denoted by H, is positive (this is equivalent to H is a tree of Eulerian graphs with $\epsilon'(H)$ even as stated in Corollary 2.6 in [10]), then there exists a family of sign-circuits $\mathcal{F}$ which covers H with length at most $\frac{4}{3}|E(G)|$.

A combination of the above two lemmas leads to the following.

Lemma 2.5 *Let* F *be a 2-factor of a 2-edge-connected cubic sign graph* (G, σ)*. If* F *contains an even number of negative edges, then there exists a family of sign-circuits* $\mathcal{F}$ *which covers* F *with length at most* $\frac{10}{9}|E(G)|$.

Proof We may assume that the 2-factor F consists of cycles $C_1, C_2, \ldots, C_t$. Denote by G^* the graph obtained from G by contracting each cycle C_i of F to a single vertex c_i.

Since F contains an even number of negative edges, the number of unbalanced cycles in F is even. Without loss of generality, we may assume that $\mathcal{C} = \{C_1, C_2, \ldots, C_{2s}\}$ is the set of unbalanced cycles of F. Let $T = \{c_1, c_2, \ldots, c_{2s}\}$ and J be a minimum T-join of G^* with respect to T. Since G is 2-edge-connected and G^* is obtained from G by contracting edges, G^* is 2-edge-connected as well. By Lemma 2.3, we have $|J| \leq \frac{1}{2}|E(G^*)| = \frac{1}{6}|E(G)|$. Consider the edge set $F \cup J$ in G, and we view it as an edge-induced subgraph of G. By the definition of T-join, we can apply Lemma 2.4(b) to $F \cup J$, i.e., there exists a family of sign-circuits $\mathcal{F}$ which covers F with length

$$\begin{aligned}\ell(\mathcal{F}) &\leq \frac{4}{3}|E(F \cup J)|\\ &= \frac{4}{3}(|E(F)| + |E(J)|)\\ &\leq \frac{4}{3}\left(\frac{2}{3}|E(G)| + \frac{1}{6}|E(G)|\right)\\ &= \frac{10}{9}|E(G)|.\end{aligned}$$

This proves the lemma. □

The proof of the following lemma is inspired by the proof of Lemma 3.7 in [13] on flows of 3-edge colorable cubic signed graphs.

Lemma 2.6 *Let C be an unbalanced cycle of a cubic signed graph (G, σ). If (G, σ) is flow-admissible and $G - E(C)$ is balanced, then (G, σ) has a family $\mathcal{F}$ of sign-circuits such that*

(1) *$E(C)$ is covered by $\mathcal{F}$, and*
(2) *the length of $\mathcal{F}$ satisfies $\ell(\mathcal{F}) \leq \frac{8}{9}|E(G)| + |E(C)|$.*

Proof Let $G' = G - E(C)$. Since G' is balanced, with some switching operations, we may assume that all edges in $E(G')$ are positive and thus $E_N(G, \sigma) \subseteq E(C)$.

Let M be a component of G'. The cycle C was divided by the vertices of M into pairwise edge-disjoint paths (called segments) whose end-vertices lie in M and all inner vertices lie in C. An end-vertex of a segment is called an attachment of M. A segment is called positive (negative, resp.) if it contains an even (odd, resp.) number of negative edges. Note that $M \cup S$ is unbalanced (balanced, resp.) if and only if the segment S is negative (positive, resp.). Since $M \cup C$ is unbalanced, the number of negative segments determined by M is odd. □

Case 1. There exists a component M of G' that determines more than one negative segments.

Then in this case M determines at least three negative segments and so $|E(C)| \geq 3$. Let $u_1Cv_1, u_2Cv_2, u_3Cv_3$ be three consecutive negative segments (in clockwise order) where u_i and v_i are attachments for $i = 1, 2, 3$. Then v_1Cu_2, v_2Cu_3 and v_3Cu_1 must be positive segments because C is unbalanced. This implies that C can be partitioned into three pieces: u_1Cu_2, u_2Cu_3, and u_3Cu_1 all contain an odd number of negative edges. Note that u_i and u_j are not adjacent in G for distinct $i, j \in \{1, 2, 3\}$ since G is cubic. Let P_1 be a (u_1, u_2)-path in M. Since M is connected, there is a path P_2 from u_3 to P_1 such that $|V(P_2) \cap V(P_1)| = 1$. Let v be the only common vertex in P_1 and P_2. Then C, P_1, P_2 form a signed graph H_1 as illustrated in Fig. 1.

Note that $|E(H_1)| \leq \frac{2}{3}|E(G)| + 2$ since G is cubic and there are exactly four vertices of degree 3 in H_1. Divide P_1 into two pieces in M: $u_1P_{11}v$ and $vP_{12}u_2$, denote by $B_1 = u_1P_{11}v \cup vP_{12}u_2 \cup u_2Cu_1, B_2 = u_3P_2v \cup vP_{11}u_1 \cup u_1Cu_3, B_3 = u_2P_{12}v \cup vP_2u_3 \cup u_3Cu_2$. Note that each of $u_2Cu_1, u_1Cu_3, u_3Cu_2$ contains an even number of negative edges. So B_1, B_2, B_3 are all balanced cycles and $\mathcal{F}_1 = \{B_1, B_2\}$, $\mathcal{F}_2 = \{B_2, B_3\}$, $\mathcal{F}_3 = \{B_3, B_1\}$ are all sign-circuit covers of H_1, which are also sign-circuit covers of C since $E(C) \subset E(H_1)$. Note that $\mathcal{B} = \{B_1, B_2, B_3\}$ is a double cover of H_1. So we have

$$\begin{aligned}\min\{\ell(\mathcal{F}_1), \ell(\mathcal{F}_2), \ell(\mathcal{F}_3)\} &\leq \frac{1}{3}(\ell(\mathcal{F}_1) + \ell(\mathcal{F}_2) + \ell(\mathcal{F}_3))\\ &= \frac{2}{3}\ell(\mathcal{B}) = \frac{4}{3}|E(H_1)|\\ &\leq \frac{4}{3}\left(\frac{2}{3}|E(G)| + 2\right)\\ &< \frac{8}{9}|E(G)| + |E(C)|.\end{aligned}$$

Case 2. Each component of G' determines exactly one negative segment.

Let $\mathcal{M}$ denote the set of all components of G'. For each component M, denote by $S_M = uCv$ the negative segment determined by M where u and v are two attachments of M on C. Denote by $S'_M = vCu$ the cosegment of S_M, which is the complement of

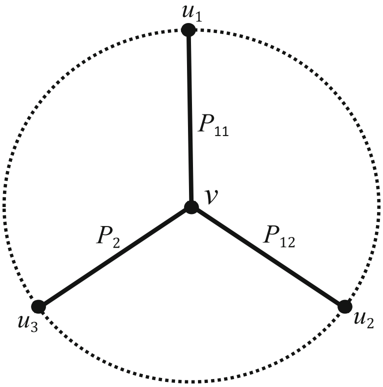


Fig. 1 $H_1 = C \cup P_{11} \cup P_{12} \cup P_2$, negative segments are dashed

S_M on C. Then $E(S_M) \neq \emptyset$ and $E(S'_M) = E(C) - E(S_M)$. We have the following two conclusions:

Claim 2.7 (*see Claim* 3.7.2 *in* [13])

$$\cap_{M\in\mathcal{M}} E(S_M) = \emptyset,$$

or equivalently, $\cup_{M\in\mathcal{M}} E(S'_M) = C$ and $|\mathcal{M}| \geq 2$.

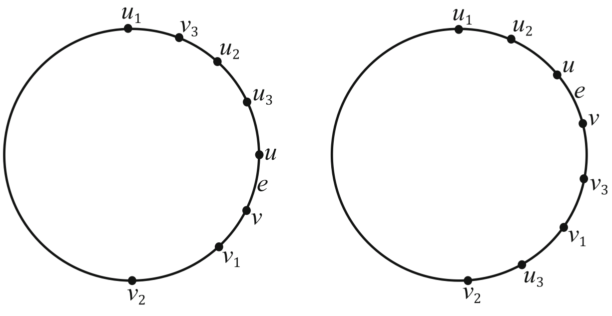
Let $\mathcal{S} = \{S'_1, S'_2, ..., S'_t\}$ be a minimal cosegment cover of C. We have

Claim 2.8 (*see Claim* 3.7.3 *in* [13]) *For any edge $e \in E(C)$, e is contained in at most two cosegments.*

Proof (Proof Sketches of Claims 1 and 2:) For the sake of completeness, we present the proof sketches of these two claims here. Suppose to the contrary $\cap_{M\in\mathcal{M}} E(S_M)$ / $= \emptyset$ and $e^* \in \cap_{M\in\mathcal{M}} E(S_M)$. Then there is a spanning tree T of $G - e^*$ containing the path $P^* = C - e^*$. Let $e = uv \in E(G) - e^* - E(T)$. Denote the unique cycle contained in $T + e$ by C_e.

If $E(C_e) \cap E(P^*) = \emptyset$, then C_e contains no negative edges and thus is balanced. Otherwise since T contains all the edges in $C - e^*$, $E(C_e) \cap E(C)$ is a path P on C. let u' and v' be the two end-vertices of P in clockwise order on C. Then $C_e - V(P) + \{u', v'\}$ is a also a path and thus it is contained in some component $M \in \mathcal{M}$. This implies that u' and v' are two attachments of M on C. Since e^* belongs to the only negative segment of C determined by M, $u'Cv'$ is the union of some positive segments of C determined by M. Therefore C_e has an even number of negative edges and thus is balanced. By Lemma 2.2, $G - e^*$ is balanced, contradicting Lemma 2.1. This proves $\cap_{M\in\mathcal{M}} E(S_M) = \emptyset$. Since $E(S'_M) = E(C) - E(S_M)$ and $\cap_{M\in\mathcal{M}} E(S_M) = \emptyset$, we have $\cup_{M\in\mathcal{M}} E(S'_M) = C$. Since $E(S_M) \neq \emptyset$ and $\cap_{M\in\mathcal{M}} E(S_M) = \emptyset$, we have $|\mathcal{M}| \geq 2$. This completes the proof of the claim 1.

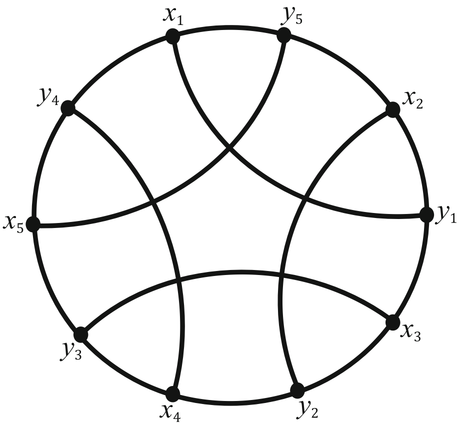
Suppose to the contrary that there exists an edge $e = uv$ that belongs to three cosegments L_1, L_2, L_3 of $\mathcal{S}$. Denote $L_i = u_iCv_i$ for each $i = 1, 2, 3$. Without loss of generality, we may assume that u_2 belongs to u_1Cu. Then v_2 doesn't belong to u_1Cv_1 (see Fig. 2). Note that v_3 belongs to u_1Cu_3. If u_3 belongs to u_1Cu, then both v_3 and u_3 belong to u_1Cv_1 and thus $L_1 \cup L_3 = C$ (see Fig. 2a), contradicting the minimality of $\mathcal{S}$.



(a) (b)

Fig. 2 Three cosegments sharing an edge

Fig. 3 Minimal cosegment cover of C with $t = 5$



If u_3 doesn't belong to u_1Cu, then u_3 belongs to vCv_2. Since L_3 contains uv, v_3 belongs to vCv_2. Thus both v_3 and u_3 belong to u_2Cv_2. Therefore $L_2 \cup L_3 = C$ (see Fig. 2b), also contradicting the minimality of $\mathcal{S}$. This completes the proof of the claim 2. □

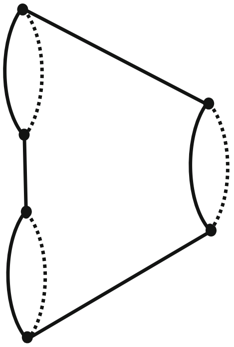
For each $i = 1, ..., t$, denote by $S_i' = x_iCy_i$ and let P_i be a path in M_i connecting x_i and y_i. Then $C_i = S_i' \cup P_i$ is a balanced Eulerian subgraph. By Claims 2.7 and 2.8, we may assume that the vertices $x_1, y_t, x_2, y_1, ..., x_t, y_{t-1}, x_1$ appear on C in clockwise order. Then $C_i \cap C_j \neq \emptyset$ if and only if $|j - i| \equiv 1(\mathrm{mod\ t})$. Moreover $|C_i \cap C_{i+1}| = x_{i+1}Cy_i$, where the sum of the indices are taken modulo t. See Fig. 3 for an illustration with $t = 5$.

Let $H_2 = C \cup P_1 \cup P_2 \cup \cdots \cup P_t$ and $B_i = x_iCy_i \cup y_iP_ix_i$ for $i = 1, 2, ..., t$. Note that B_i is a balanced cycle and so $\mathcal{F} = \{B_1, B_2, ..., B_t\}$ is a sign-circuit cover of C. Obviously, $|E(H_2)| \leq \frac{2}{3}|E(G)| + t$ since G is cubic and there are exactly $2t$ vertices of degree 3 in H_2. By Claim 2.8, $\mathcal{F}$ covers the edges in C at most twice and edges in $P_1 \cup \cdots \cup P_t$ exactly once. Let W be the set of edges covered by $\mathcal{F}$ twice. Then $W \subset E(C)$. Since G is cubic, $x_i \neq y_j$ for all $i, j \in \{1, 2, ..., t\}$. So we have $|W| \leq |E(C)| - t$. Therefore,

$$\begin{aligned}\ell(\mathcal{F}) &= |E(B_1)| + |E(B_2)| + \cdots + |E(B_t)| \\ &\leq |E(H_2)| + |E(C)| - t \\ &\leq \frac{2}{3}|E(G)| + t + |E(C)| - t \\ &\leq \frac{8}{9}|E(G)| + |E(C)|.\end{aligned}$$

This completes the proof. □

Fig. 4 (G, σ) with a shortest sign-circuit cover with length $\frac{13}{9}|E(G)|$



3 Proof of Theorem 1.4

Proof of Theorem 1.4 Let f be a 3-edge coloring of connected cubic graph G. Let R, B, Y be the three color classes of f. Recall that $E_N(G, \sigma)$ is the set of negative edges in (G, σ). Without loss of generality, we may assume $|R \cap E_N(G, \sigma)| \equiv |B \cap E_N(G, \sigma)| (\text{mod } 2)$. Denote by M_1M_2 the 2-factor induced by $M_1 \cup M_2$ for each pair $M_1, M_2 \in \{R, B, Y\}$. Since $|R \cap E_N(G, \sigma)| \equiv |B \cap E_N(G, \sigma)| (\text{mod } 2)$, RB has an even number of unbalanced components. By Lemma 2.5, there exists a family of sign-circuits $\mathcal{F}_1$ which covers RB with length at most $\frac{10}{9}|E(G)|$.

Case 1. *RB contains an unbalanced cycle.*

First, assume that $|Y \cap E_N(G, \sigma)|$ has the same parity with $|R \cap E_N(G, \sigma)|$. Then RY has an even number of unbalanced cycles. By Lemma 2.5, we can find a family of sign-circuits $\mathcal{F}_2$ which covers RY with length at most $\frac{10}{9}|E(G)|$. Therefore, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is a sign-circuit cover of G with length

$$\ell(\mathcal{F}) = \ell(\mathcal{F}_1) + \ell(\mathcal{F}_2) \leq \frac{10}{9}|E(G)| + \frac{10}{9}|E(G)| = \frac{20}{9}|E(G)|.$$

Then, assume instead that $|Y \cap E_N(G, \sigma)|$ has different parity with $|R \cap E_N(G, \sigma)|$. Let C be an unbalanced cycle in RB. Now we swap the colors R and B on C, i.e. reset $R' = R \triangle E(C)$ and $B' = B \triangle E(C)$ respectively. Note that the operation will change the parity of $|R \cap E_N(G, \sigma)|$ and $|B \cap E_N(G, \sigma)|$. This implies that $|Y \cap E_N(G, \sigma)| \equiv |R' \cap E_N(G, \sigma)| \equiv |B' \cap E_N(G, \sigma)| (\text{mod } 2)$ now. So, similar as the previous paragraph, we apply Lemma 2.5 to find a family of sign-circuits $\mathcal{F}_2'$ which covers $R'Y$ with length at most $\frac{10}{9}|E(G)|$. Notice that $\mathcal{F}_1$ covers $RB = R'B'$ with length at most $\frac{10}{9}|E(G)|$. Hence $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2'$ is a sign-circuit cover of G with length at most $\frac{20}{9}|E(G)|$.

Case 2. *RB contains no unbalanced cycle.*

In this case, each cycle C_i of RB is a balanced cycle. Let $\mathcal{F}_1 = \{C_i | C_i \text{ is a balanced cycle of } RB\}$. Then $\mathcal{F}_1$ is a family of sign-circuits covering RB with length $\ell(\mathcal{F}_1) = |E(RB)| = \frac{2}{3}|E(G)|$.

Subcase 2.1: The number of unbalanced cycles in RY or BY is even.

By Lemma 2.5, we have a family of sign-circuits $\mathcal{F}_2$ which covers RY or BY with length at most $\frac{10}{9}|E(G)|$. Therefore $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is a sign-circuit cover of G with length $\ell(\mathcal{F}) \leq \frac{16}{9}|E(G)|$.

Subcase 2.2: The number of unbalanced cycles in RY or BY is equal to one.

Without loss of generality, assume that RY has exactly one unbalanced component, say, C_1. Let $\mathcal{C} = \{C_1, \ldots, C_m\}$ be the set of components of RY, where each C_i $(i \geq 2)$ is balanced. Let $\mathcal{F}_2 = \{C_i : i \geq 2\}$. Then $\mathcal{F}_2$ is a family of sign-circuits covering $RY - E(C_1)$ with length $\frac{2}{3}|E(G)| - |E(C_1)|$. We consider the following two cases in order to cover C_1.

Assume first that G contains an unbalanced circuit C' with $E(C') \cap E(C_1) = \emptyset$. Since G is cubic and connected, there is a long barbell Q in G with P as the path connecting C_1 and C' with $|E(Q)| \leq \frac{2}{3}|E(G)| + 1$.

Therefore, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup Q$ is a sign-cycle cover of G with length

$$\begin{aligned}\ell(\mathcal{F}) &= \ell(\mathcal{F}_1) + \ell(\mathcal{F}_2) + \ell(Q)\\ &\leq \frac{2}{3}|E(G)| + \frac{2}{3}|E(G)| - |E(C_1)| + \frac{2}{3}|E(G)| + 1\\ &< 2|E(G)|.\end{aligned}$$

Then assume instead that G contains no unbalanced cycle C' with $E(C') \cap E(C_1) = \emptyset$. In this case, $G - E(C_1)$ is balanced. By Lemma 2.6, there exists a family $\mathcal{F}_3$ of sign-circuits covering $E(C_1)$ with length at most $\frac{8}{9}|E(G)| + |E(C_1)|$. Therefore, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ is a sign-circuit cover of G with length

$$\begin{aligned}\ell(\mathcal{F}) &= \ell(\mathcal{F}_1) + \ell(\mathcal{F}_2) + \ell(\mathcal{F}_3)\\ &\leq \frac{2}{3}|E(G)| + \frac{2}{3}|E(G)| - |E(C_1)| + \frac{8}{9}|E(G)| + |E(C_1)|\\ &\leq \frac{20}{9}|E(G)|.\end{aligned}$$

Subcase 2.3: *The number of unbalanced cycles in each of RY, BY is odd and is at least 3.*

Let $\mathcal{C} = \{C_1, \ldots, C_m\}$ be the set of components of RY. Denote by G^* the graph obtained from G by contracting each cycle C_i of RY to a single vertex u_i, where $i = 1, 2, \ldots, m$. Note that G^* is connected, and so let T^* be a spanning tree of G^*. Then $T^* \cup RY$ is a cycle-tree in G, denoted by H, containing at least 3 unbalanced cycles. Let B' be the set of bridges of H such that $b_i \in B'$ if and only if $(H - b_i, \sigma|_{H-b_i})$ has a balanced component H_i. $B = \emptyset$ if no such bridge exists in H. Let $H' = H - (B' \cup (\cup_{i=1,\ldots,|B'|} E(H_i)))$. Note that every leaf-cycle of H' is unbalanced and H' contains all the unbalanced cycles of $\mathcal{C}$. By Lemmas 2.1 and 2.4(a), H' is flow-admissible and has a sign-circuit cover $\mathcal{F}_2$ with length at most $\frac{3}{2}|E(H')|$. Since the cycles in H_i are all balanced cycles, we can cover them with length at most $|\cup_{i=1,\ldots,|B'|} E(H_i)| \leq |E(H)| - |E(H')|$. Thus we have a sign-circuit cover $\mathcal{F}_3$ of RY with length

$$\begin{aligned}\ell(\mathcal{F}_3) &= \ell(\mathcal{F}_2) + (|E(H)| - |E(H')|)\\ &\le \frac{3}{2}|E(H')| + |E(H)| - |E(H')|\\ &\le \frac{3}{2}|E(H)| < \frac{3}{2}|E(G)|.\end{aligned}$$

Therefore, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_3$ is a sign-circuit cover of G with length

$$\begin{aligned}\ell(\mathcal{F}) &= \ell(\mathcal{F}_1) + \ell(\mathcal{F}_3)\\ &\le \frac{2}{3}|E(G)| + \frac{3}{2}|E(G)|\\ &= \frac{13}{6}|E(G)| < \frac{20}{9}|E(G)|.\end{aligned}$$

This completes the proof. □

Remark The upper bound of $scc(G)$ in Theorem 1.4 seems not to be tight. We realized that the signed 3-edge-colorable cubic graph (G, σ) as illustrated in Fig. 4 has a sign-circuit cover with length $\frac{13}{9}|E(G)|$. The problem to determine the optimal upper bound for the shortest sign-circuit cover of signed 3-edge-colorable cubic graph remains open.

Data availability statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest No conflict of interest exits in the submission of this manuscript, and manuscript is approved by all authors for publication. I would like to declare on behalf of my co-authors that the work described was original research that has not been published previously, and not under consideration for publication elsewhere, in whole or in part. All the authors listed have approved the manuscript that is enclosed.

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