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Modulo k-orientations of random regular graphs

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Dedicated to Professor Fuji Zhang on the occasion of his 90th birthday

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ABSTRACT

A modulo k-orientation of a graph is an orientation where the in-degree is congruent to the out-degree modulo k for each vertex. Jaeger conjectured in 1981 that all $4p$-edge-connected graphs admit modulo $(2p+1)$-orientations for any integer $p \geq 1$. This conjecture is equivalent to its restriction to $(4p+1)$-regular graphs by Mader's splitting lemma. In 2018, Jaeger's conjecture was disproved for all $p \geq 3$, but it remains open for $p = 1$ and $p = 2$, corresponding to Tutte's 3-flow conjecture and 5-flow conjecture, respectively. In 2020, Prałat and Wormald (Prałat and Wormald, 2020) proved that random 5-regular graphs asymptotically almost surely have modulo 3-orientations. This paper extends that result by proving that random $(4p+1)$-regular graphs asymptotically almost surely have modulo $(2p+1)$-orientations for $p \leq 4$, which serves as a random graph analog to Jaeger's conjecture. Our proof uses the Small Subgraph Conditioning Method for random regular graphs and reveals that this method fails when $p \geq 5$.

1. Introduction

Tutte [20] introduced the concept of nowhere-zero flows as a generalization of map coloring problems. As a potential extension of Grötzsch's theorem, Tutte in 1972 conjectured that every 4-edge-connected graph admits a nowhere-zero 3-flow (see [4]). A graph has a nowhere-zero 3-flow if and only if it has an orientation such that each vertex has in-degree congruent to its out-degree modulo 3, which is called a *modulo 3-orientation*. More generally, a *modulo k-orientation* of a graph G is an orientation such that every vertex has in-degree congruent to out-degree modulo k. In 1981, Jaeger [10,11] generalized Tutte's conjecture by proposing the Circular Flow Conjecture that every $4p$-edge-connected graph admits a modulo $(2p+1)$-orientation. For $p = 1$, it is equivalent to Tutte's 3-flow conjecture, and for $p = 2$, the validity of this special case implies the truth of Tutte's 5-flow conjecture (see [22]). Lovász, Thomassen, Wu and Zhang [15] in 2013 confirmed Jaeger's Circular Flow Conjecture for all $6p$-edge-connected graphs. However, Han, Li, Wu and Zhang [9] in 2018 disproved Jaeger's Circular Flow Conjecture for each $p \geq 3$ by constructing infinite families of counterexamples. By applying Mader's splitting lemma [16], Jaeger's Circular Flow Conjecture can be reduced to its restriction to $(4p+1)$-regular graphs. For $p = 1$, Prałat and Wormald [18] in 2020 were able to confirm this conjecture for random 5-regular graphs asymptotically almost surely.

In the random graph model, we say *almost all* graphs have some property A_n in a probability space if $\mathbb{P}(A_n)$ converges to 1 as n goes to infinity. We also call it by means of *with high probability*, *almost surely*, *asymptotically almost surely* and so on. For the sake of calculation, we would always use the *pairing model* instead of general *uniform random regular graph model*, see [12] or [21] for more details. The result of Prałat and Wormald is formally stated as follows.

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Theorem 1.1 (*Prałat and Wormald [18]*)**.** *Almost all* 5*-regular graphs admit modulo* 3*-orientations.*

For general $(4p+1)$-regular graphs, Alon and Prałat [1] in 2011 showed that for sufficiently large p, almost all random $(4p+1)$-regular graphs have modulo $(2p+1)$-orientations. Note that Alon and Prałat [1] did not provide an explicit lower bound on p. However, from their proof, it appears that their method suggests a bound of approximately $p \geq 10^{13}$.

Theorem 1.2 (*Alon and Prałat [1]*)**.** *For $p \geq 10^{13}$, a random $(4p+1)$-regular graph G admits a modulo $(2p+1)$-orientation asymptotically almost surely.*

Even given the counterexamples of [9], it would be of interest to explore whether for each p almost all random $(4p+1)$-regular graphs have modulo $(2p+1)$-orientations, and this question remains open for only finitely many $p < 10^{13}$. The case $p = 2$ is related to Tutte's 5-flow conjecture, and Delcount, Huq and Prałat [7], and J. Liu [14], independently, investigated this case and provided a positive answer.

Note that the case $p = 3$ is also special as it is the first known p for which we know there exist counterexamples to Jaeger's Circular Flow Conjecture. In this paper, we confirm this conjecture for a random 13-regular graph even though there exist infinitely many counterexamples. This indicates that almost all 13-regular graphs satisfy Jaeger's Circular Flow Conjecture in the probability sense. Moreover, we prove the following main result.

Theorem 1.3. *For any positive integer p with $p \leq 4$, a random $(4p+1)$-regular graph admits a modulo $(2p+1)$-orientation asymptotically almost surely.*

Our proof applies the Small Subgraph Conditioning Method, introduced by Robinson and Wormald [19] and later developed by Janson [12]. Specifically, this paper proves that for $p = 2, 3, 4$, almost all $(4p+1)$-regular graphs have modulo $(2p+1)$-orientation by using Small Subgraph Conditioning Method, abbreviated by *SSCM*.

Theorem 1.4 (*Small Subgraph Conditioning Method, See [12,21]*)**.** *Let $\lambda_i > 0$ and $\delta_i \geq -1$, $i = 1, 2, \ldots$, be constants and suppose that for each n there are random variables X_{in}, $i = 1, 2, \ldots$ and Y_n(defined on the same probability space) such that X_{in} is non-negative integer-valued and $\mathbb{E}Y_n \neq 0$(at least for large n), and furthermore, the following conditions are satisfied:*

1. *$X_{in} \xrightarrow{d} Z_i$ as $n \to \infty$, jointly for all i, where Z_i ∼Poi(λ_i) are independent Poisson random variables with means λ_i;*
2. *For any finite sequence $j_1, \ldots, j_m$ of non-negative integers,*

$$\frac{\mathbb{E}[Y_n[X_{1n}]_{j_1} \ldots [X_{mn}]_{j_m}]}{\mathbb{E}Y_n} \to \prod_{i=1}^{m}(\lambda_i(1+\delta_i))^{j_i}$$

as $n \to \infty$, where $[x]_j = x(x-1)\ldots(x-j+1)$;

3. *$\sum_{i\geq 1} \lambda_i \delta_i^2 < \infty$;*
4. *$\frac{\mathbb{E}Y_n^2}{(\mathbb{E}Y_n)^2} \to \exp(\sum_{i\geq 1} \lambda_i \delta_i^2)$ as $n \to \infty$.*

Then $W > 0$ asymptotically almost surely if and only if $\delta_i > -1$ for all i.

Many results have been proved by the Small Subgraph Conditioning Method. For example, in 2018, Postle and Delcount [8] proved almost all 4-regular graphs have a 3-stars decomposition. In 2023, Delcourt, Greenhill, Isaev, Lidický and Postle [6] proved a further result. More examples can be found in [2,5,13].

When verifying the condition (3) of the Small Subgraph Conditioning Method, a straightforward calculation suggests that it fails when $p \geq 5$. Hence there is a relevant problem as follows.

Problem 1.5. For any integer $p \geq 5$, do almost all random $(4p+1)$-regular graphs admit modulo $(2p+1)$-orientations?

Note that this problem is solved when p is large as can be seen in Theorem 1.2, and there are only finitely many p remaining open and requiring some new ideas.

2. Proof of Theorem 1.3

We shall divide our proof into several subsections. In Section 2.1, we calculate the expectation of Y defined below. In Section 2.2, we verify the conditions (2) and (3) of Theorem 1.4. In Section 2.3, we calculate the second moment of Y and verify the condition (4) of Theorem 1.4. We relegate the tedious calculation of the Hessian matrix determinant to Appendix B.

Through the proof, we assume that a graph (or a pairing) has n vertices where n is even. For simplicity, we omit the subscript 'n' for whenever there is no ambiguity. Let Y denote the number of modulo $(2p+1)$-orientations in a random $(4p+1)$-regular graph. Thus we just need to prove $Y > 0$ for almost all random $(4p+1)$-regular graphs.

Next we shall verify the four conditions of Theorem 1.4. In the pairing model, each $(4p+1)$-*regular pairing* contains n vertices, and each vertex can be regarded as a *cell* consists of $4p+1$ *points* (In this paper, we shall apply the both terminology "vertex" and "cell"). Clearly, matching these $(4p+1)n$ points forms a $(4p+1)$-regular multigraph. Let $M((4p+1)n)$ denote the number of $(4p+1)$-regular pairings on n vertices. The number of ways to match $(4p+1)n$ points equals

$$M((4p+1)n) = \frac{((4p+1)n)!}{\left(\frac{(4p+1)n}{2}\right)!2^{\frac{(4p+1)n}{2}}}.$$

For a modulo $(2p+1)$-orientation in a $(4p+1)$-regular pairing, we can assign each point as *in-point* or *out-point* before matching them to form a directed $(4p+1)$-regular multigraph with modulo $(2p+1)$-orientation directly. To be specific, all the cells in a $(4p+1)$-regular pairing can be partitioned into two parts with the same size. The number of out-points minus the number of in-points of cell in one part is equal to $2p+1$, we refer to the corresponding vertex as an *out-vertex*, and we refer to the p in-points in the cell as *out-vertex special points*. In the other part, the number of out-points minus the number of in-points of cell is equal to $-(2p+1)$, we refer to the corresponding vertex as an *in-vertex*, and we refer to the p out-points in the cell as *in-vertex special point*. For convenience, we refer to both the out-vertex special points and in-vertex special points as *special points*. In a directed cycle, we refer to a vertex v as *source* (or *sink*, respectively) if its two incident edges in the cycle are oriented outwards from v (towards v, respectively).

2.1. The expectation

In this subsection, we estimate the expectation of Y as follows.

Theorem 2.1. *We have* $\mathbb{E}[Y] \sim \sqrt{4p+1}\left(\frac{\sqrt{2}\binom{4p+1}{p}}{2^{2p}}\right)^n$.

Proof. We first partition vertex set of a pairing into two parts with the same size, and vertices are assigned to in-vertex in one part and out-vertex in the other part. Then select the p special points in each cell. At this time, each point in pairing is assigned to in-point or out-point. Matching all the points such that each in-point is matched with an out-point gives the total number of modulo $(2p+1)$-orientations that all the $(4p+1)$-regular pairings have. Multiplying the above factors and dividing by the number of pairings gives the expectation of Y. Finally by Stirling's approximation, we have

$$\begin{aligned}
\mathbb{E}[Y] &= \frac{1}{M((4p+1)n)}\binom{n}{n/2}\binom{4p+1}{p}^n\left(\frac{(4p+1)n}{2}\right)! \\
&= \frac{\left(\frac{(4p+1)n}{2}\right)!2^{\frac{(4p+1)n}{2}}}{((4p+1)n)!}\frac{n!}{(n/2)!(n/2)!}\binom{4p+1}{p}^n\left(\frac{(4p+1)n}{2}\right)! \\
&\sim \sqrt{4p+1}\left[\frac{\binom{4p+1}{p}^2}{2^{4p-1}}\right]^{\frac{n}{2}}. \quad \square
\end{aligned}$$

2.2. The cycle conditioning

When applying the Small Subgraph Conditioning Method, we choose cycles as the small subgraphs—these correspond to the random variables X_i in Theorem 1.4. The following lemma gives asymptotic distribution of cycles in random regular graph, which suggests that condition (1) of Theorem 1.4 holds.

Lemma 2.2 (See [3], Theorem 2.16)**.** *For d fixed, let X_j denote the number of cycles of length j in the random multigraph resulting from a pairing. For $j \geq 1$, $X_1, \dots, X_j$ are asymptotically independent Poisson random variables with means $\lambda_j = \frac{(d-1)^j}{2j}$.*

The following lemma gives the value of δ_i and verifies the condition (2) of Theorem 1.4.

Lemma 2.3. *Let Y denote the number of modulo $(2p+1)$-orientations in a random $(4p+1)$-regular graph, and let X_j denote the number of cycles of length j in the graph. Then, we have the asymptotic relation:*

$$\frac{\mathbb{E}[YX_j]}{\mathbb{E}[Y]} \sim \frac{(4p)^j}{2j}\left(1+(-\frac{p}{4p+1})^j\right).$$

Before the proof, as listed in Table 1, notice that this lemma suggests that we can take $\delta_j = (-\frac{p}{4p+1})^j$ in Theorem 1.4. Then we can verify the condition (3) of Theorem 1.4, and obtain that

$$\begin{aligned}
\sum_{j=1}^{\infty}\lambda_j\delta_j^2 &= \sum_{j\geq 1}\frac{(4p)^j}{2j}\left(-\frac{p}{4p+1}\right)^{2j} = \sum_{j\geq 1}\frac{1}{2j}\left(\frac{4p^3}{(4p+1)^2}\right)^j \\
&= -\frac{1}{2}\ln(1-x)\Big|_{x=\frac{4p^3}{(4p+1)^2}} = \frac{1}{2}\ln\left[\frac{(4p+1)^2}{(4p+1)^2-4p^3}\right].
\end{aligned}$$

Table 1
the values of δ_j and its corresponding expectation

p	δ_j	$\exp\left(\sum_i \lambda_i \delta_i^2\right)$
2	$(-\frac{2}{9})^j$	$\frac{9}{7}$
3	$(-\frac{3}{13})^j$	$\frac{13}{\sqrt{61}}$
4	$(-\frac{4}{17})^j$	$\frac{17}{\sqrt{33}}$
5	$(-\frac{5}{21})^j$	∞

The calculation gives that when $p \geq 5$, $\frac{4p^3}{(4p+1)^2} > 1$, which indicates that relevant progression does not converge and Small Subgraph Conditioning Method fails. For exact p, when $p = 2$, $\exp(\sum_i \lambda_i \delta_i^2) = \frac{9}{7}$; When $p = 3$, $\exp(\sum_i \lambda_i \delta_i^2) = \frac{13}{\sqrt{169-108}} = \frac{13}{\sqrt{61}}$; When $p = 4$, $\exp(\sum_i \lambda_i \delta_i^2) = \frac{17}{\sqrt{289-256}} = \frac{17}{\sqrt{33}}$ (see Table 1).

Proof of Lemma 2.3. Actually, the proof of this Lemma is a counting process. Since $\mathbb{E}[YX_j]$ and $\mathbb{E}[Y]$ are in the same probability space, we only need to divide the total number of YX_j by the total number of Y.

1. The number of all possible cycles of length j in an n-vertex graph is $\frac{1}{2}\binom{n}{j}(j-1)!$ and the number of orientations of such a cycle with exactly s sinks and s sources is $2\binom{j}{2s}$.
2. For s sinks in the cycle mentioned above, suppose that there are exactly x_1 out-vertex and $(s - x_1)$ in-vertex among them. For s sources, suppose that there are exactly x_2 out-vertex and $(s - x_2)$ in-vertex among them. For the other $(j - 2s)$ vertices in the cycle, suppose that there are x_3 out-vertex and $(j - 2s - x_3)$ in-vertex. For $(n - j)$ vertices other than cycle, there are $(\frac{n}{2} - x_1 - x_2 - x_3)$ out-vertex and $(\frac{n}{2} + x_1 + x_2 + x_3 - j)$ in-vertex.
3. After partitioning vertices into in-vertex set and out-vertex set, we then determine the in-points and out-points in each cell, and match them.
4. Finally, we divide the product of these factors by the number of all modulo $(2p+1)$-orientations in $(4p+1)$-regular pairings.

Since

$$\frac{\binom{n-j}{\frac{n}{2}-x_1-x_2-x_3}}{\binom{n}{\frac{n}{2}}} = \frac{(n-j)!}{n!}\frac{\frac{n}{2}!\frac{n}{2}!}{(\frac{n}{2}-x_1-x_2-x_3)!(\frac{n}{2}+x_1+x_2+x_3-j)!} \sim \frac{1}{2^j}$$

and take $\lambda_j = \frac{(4p)^j}{2j}$ and $\delta_j = (-\frac{p}{4p+1})^j$, we obtain that

$$\begin{aligned}
\frac{\mathbb{E}[YX_j]}{\mathbb{E}[Y]} &= \frac{1}{\binom{n}{n/2}\binom{4p+1}{p}^n \frac{(4p+1)n}{2}!}\frac{1}{2}\binom{n}{j}(j-1)!(4p(4p+1))^j \\
&\sum_{s=0}^{\lfloor \frac{j}{2} \rfloor} 2\binom{j}{2s}\sum_{x_1=0}^{s}\binom{s}{x_1}\binom{4p-1}{p}^{x_1}\binom{4p-1}{p-2}^{s-x_1}\sum_{x_2=0}^{s}\binom{s}{x_2}\binom{4p-1}{p-2}^{x_2}\binom{4p-1}{p}^{s-x_2} \\
&\sum_{x_3=0}^{j-2s}\binom{j-2s}{x_3}\binom{4p-1}{p-1}^{j-2s}\binom{n-j}{\frac{n}{2}-x_1-x_2-x_3}\binom{4p+1}{p}^{n-j}\left(\frac{(4p+1)n}{2}-j\right)! \\
&\sim \sum_{s=0}^{\lfloor \frac{j}{2} \rfloor}\frac{(4p(4p+1))^j n^j\binom{j}{2s}\left(\binom{4p-1}{p}+\binom{4p-1}{p-2}\right)^{2s}\binom{4p-1}{p-1}^{j-2s}}{2^{2s} j\binom{4p+1}{p}^j(\frac{(4p+1)n}{2})^j} \\
&= \frac{(4p)^j}{j}\sum_{s=0}^{\lfloor \frac{j}{2} \rfloor}\binom{j}{2s}\left(\frac{\binom{4p-1}{p}+\binom{4p-1}{p-2}}{2\binom{4p-1}{p-1}}\right)^{2s}\left(\frac{2\binom{4p-1}{p-1}}{\binom{4p+1}{p}}\right)^j \\
&= \frac{(4p)^j}{2j}\left(\left(1+\frac{\binom{4p-1}{p}+\binom{4p-1}{p-2}}{2\binom{4p-1}{p-1}}\right)^j+\left(1-\frac{\binom{4p-1}{p}+\binom{4p-1}{p-2}}{2\binom{4p-1}{p-1}}\right)^j\right)\left(\frac{2\binom{4p-1}{p-1}}{\binom{4p+1}{p}}\right)^j \\
&= \lambda_j\left(1+\left(\frac{-p}{4p+1}\right)^j\right)
\end{aligned}$$

$$= \lambda_j(1+\delta_j).$$

This completes the proof of Lemma 2.3. □

Notice that the calculation of

$$\frac{\mathbb{E}[Y_n[X_{1n}]_{j_1}\dots[X_{mn}]_{j_m}]}{\mathbb{E}Y_n}$$

in the condition (2) of Theorem 1.4 is similar to the calculation of $\frac{\mathbb{E}[YX_j]}{\mathbb{E}[Y]}$ but more tedious. Thus we omit the details.[1]

2.3. The second moment

The primary objective of this section is to calculate the second moment of Y and prove the following theorem, which is exactly the condition (4) of Theorem 1.4.

Theorem 2.4. *When $p = 2, 3, 4$, the following asymptotic relation holds:*

$$\frac{\mathbb{E}[Y^2]}{\mathbb{E}[Y]^2} \sim \exp\big(\sum_i \lambda_i\delta_i^2\big) = \frac{4p+1}{\sqrt{(4p+1)^2-4p^3}}.$$

Calculation of the second moment is the most intricate step. We shall prove Theorem 2.4 by showing Lemmas 2.5, 2.8, Theorem 2.9 and calculating the determinant of a Hessian matrix below. Let $\phi(x) = x\ln x$ and

$$\begin{aligned}
g(\mathbf{z}) &= g(z, z_{100}, \dots, z_{p00}, z_{111}, \dots, z_{p11}, z_{101}, \dots, z_{p01}, z_{110}, \dots, z_{p10})\\
&= \frac{1}{(2\pi n)^{\frac{4p+1}{2}}}\sqrt{\frac{p+z+\sum_{i=1}^p i(z_{i00}+z_{i11}-z_{i01}-z_{i10})+\frac{1}{6n}}{2(z-\sum_{i=1}^p z_{i00}+\frac{1}{6n})(z-\sum_{i=1}^p z_{i11}+\frac{1}{6n})}}\times\\
&\quad\sqrt{\frac{p+\frac12-z-\sum_{i=1}^p i(z_{i00}+z_{i11}-z_{i01}-z_{i10})+\frac{1}{6n}}{(\frac12-z-\sum_{i=1}^p z_{i01}+\frac{1}{6n})(\frac12-z-\sum_{i=1}^p z_{i10}+\frac{1}{6n})\prod_{ijl:i\ge1}(z_{ijl}+\frac{1}{6n})}},
\end{aligned}$$

$$\begin{aligned}
f(\mathbf{z}) &= f(z, z_{100}, \dots, z_{p00}, z_{111}, \dots, z_{p11}, z_{101}, \dots, z_{p01}, z_{110}, \dots, z_{p10})\\
&= -\frac{4p+1}{2}\ln(4p+1) + \sum_{i=1}^p\sum_{(k,l)} z_{ikl}\ln\binom{4p+1}{i}\binom{4p+1-i}{p-i}\binom{3p+1}{p-i}\\
&\quad + (1-\sum_{ijl,i\ge1} z_{ijl})\ln\binom{4p+1}{p}\binom{3p+1}{p}\\
&\quad + \phi(p+z+\sum_{i=1}^p i(z_{i00}+z_{i11}-z_{i01}-z_{i10}))\\
&\quad + \phi(p+\frac12-z-\sum_{i=1}^p i(z_{i00}+z_{i11}-z_{i01}-z_{i10})) - \sum_{ijl,i\ge1}\phi(z_{ijl})\\
&\quad - \phi(z-\sum_{i=1}^p z_{i00}) - \phi(z-\sum_{i=1}^p z_{i11})\\
&\quad - \phi(\frac12-z-\sum_{i=1}^p z_{i10}) - \phi(\frac12-z-\sum_{i=1}^p z_{i01}),
\end{aligned}$$

$$\begin{aligned}
L = \Bigg\{\Big(&\frac{k}{n}, \frac{k_{100}}{n}, \dots, \frac{k_{p00}}{n}, \frac{k_{111}}{n}, \dots, \frac{k_{p11}}{n}, \frac{k_{101}}{n}, \dots, \frac{k_{p01}}{n}, \frac{k_{110}}{n}, \dots, \frac{k_{p10}}{n}\Big):\\
&k,\ k_{i00},\ k_{i01},\ k_{i10}\, k_{i11} \in \mathbb{N},\ 0 \le \sum_{i=1}^p k_{i00} \le k,\ 0 \le \sum_{i=1}^p k_{i11} \le k,\\
&0 \le \sum_{i=1}^p k_{i01} \le \frac{n}{2} - k,\ 0 \le \sum_{i=1}^p k_{i10} \le \frac{n}{2} - k,\ 0 \le k \le \frac{n}{2}\Bigg\},
\end{aligned}$$

[1] For the readers, we present this calculation in Appendix A.

where $k_{ijl} = z_{ijl}n$. The following lemma gives the expression of the second moment of Y.

Lemma 2.5. *The asymptotic relation holds:* $\mathbb{E}[Y^2] \sim \sum_{\mathbf{z}\in L} S(\mathbf{z})g(\mathbf{z})e^{f(\mathbf{z})n}$.

Proof. For a $(4p+1)$-regular pairing and its modulo $(2p+1)$-orientation, its vertices can be partitioned into two parts with the same size, and vertices in one part have $(3p+1)$ in-degree, p out-degree, while vertices in the other part have $(3p+1)$ out-degree, p in-degree. Note that $\mathbb{E}[Y^2]$ gives the expectation of the number of ways to consecutively choose two such modulo $(2p+1)$-orientations in a $(4p+1)$-regular pairing. Let k be the number of vertices that are in-vertices in both orientations, and there are $(\frac{n}{2}-k)$ vertices assigned to in-vertex in the first orientation and out-vertex in the second orientation. Let 1 represent "out", 0 represent "in", and let $j, l \in \{0, 1\}$, we say a cell is (j, l)-*type* if its related vertex is j-vertex in the first orientation and l-vertex in the second orientation. For $i \in \{0, 1, \ldots, p\}$, $j, l \in \{0, 1\}$, let k_{ijl} denote the number of (j, l)-type cells that have exactly i same special points. By above definition, we can obtain the following identities:

$$\sum_{ijl} k_{ijl} = n,$$

$$\sum_{i=0}^{p} k_{i00} = \sum_{i=0}^{p} k_{i11} = k, \tag{1}$$

$$\sum_{i=0}^{p} k_{i01} = \sum_{i=0}^{p} k_{i10} = \frac{n}{2} - k. \tag{2}$$

Then we derive the expression of $\mathbb{E}[Y^2]$.

1. The number of ways to partition vertex set according to the different types in two orientations is $\frac{n!}{\prod_{ijl} k_{ijl}!}$.
2. The number of ways to assign special points in according cell of each vertex is

$$\prod_{i=0}^{p}\left(\binom{4p+1}{i}\binom{4p+1-i}{p-i}\binom{3p+1}{p-i}\right)^{\sum_{(j,l)} k_{ijl}}.$$

3. Now we try to match all the points such that every in-point is matched with an out-point in both orientations. Thus the size of $(1, 1) \leftrightarrow (0, 0)$ matching is exactly the number of $(0, 0)$ points, that is

$$\sum_{i=0}^{p} ik_{i00} + \sum_{i=0}^{p}(2p+1+i)k_{i11} + \sum_{i=0}^{p}(p-i)k_{i01} + \sum_{i=0}^{p}(p-i)k_{i10}$$
$$= pn + k + \sum_{i=1}^{p} i(k_{i00} + k_{i11} - k_{i01} - k_{i10}).$$

Similarly, the size of $(1, 0) \leftrightarrow (0, 1)$ matching is exactly the number of $(1, 0)$ points, that is

$$\frac{2p+1}{2}n - k - \sum_{i=1}^{p} i(k_{i00} + k_{i11} - k_{i01} - k_{i10}).$$

Thus

$$\mathbb{E}[Y^2] = \frac{\frac{(4p+1)n}{2}!\,2^{\frac{(4p+1)n}{2}}}{((4p+1)n)!} \sum_{k,k_{ijl}} \frac{n!}{\prod_{ijl} k_{ijl}!} \prod_{i=0}^{p}\left(\binom{4p+1}{i}\binom{4p+1-i}{p-i}\binom{3p+1}{p-i}\right)^{\sum_{(j,l)} k_{ijl}}$$
$$(pn + k + \sum_{i=1}^{p} i(k_{i00} + k_{i11} - k_{i01} - k_{i10}))!$$
$$(\frac{2p+1}{2}n - k - \sum_{i=1}^{p} i(k_{i00} + k_{i11} - k_{i01} - k_{i10}))!$$
$$\sim \sum_{\mathbf{z}\in L} S(\mathbf{z})g(\mathbf{z})e^{f(\mathbf{z})n}.$$

The last approximation is obtained by applying (1)(2) to erase k_{0jl} and Gosper's formula

$$s! = (1 + O(\frac{1}{s}))\sqrt{\pi(2s + \frac{1}{3})}(\frac{s}{e})^s,$$

and let $zn = k$, $z_{ijl}n = k_{ijl}$. In our final expression, $g(\mathbf{z})$, $f(\mathbf{z})$, L are defined above, and $S(\mathbf{z})$ is the error term arising from Gosper's formula, $g(\mathbf{z})$ is the polynomial term, $f(\mathbf{z})$ is the exponential term. So the relation holds. □

Note that $k_{000}, k_{001}, k_{010}, k_{011}$ can be determined by other variables in the range L and thus we do not regard them as independent variables. When n goes to infinity, the closure of L becomes the domain of f, denoted by

$$\begin{aligned} R =\{&(z, z_{100}, \dots, z_{p00}, z_{111}, \dots, z_{p11}, z_{101}, \dots, z_{p01}, z_{110}, \dots, z_{p10}) : \\ &z,\ z_{i00},\ z_{i01},\ z_{i10}\, z_{i11} \in \mathbb{R}^+,\ 0 \le \sum_{i=1}^{p} z_{i00} \le z,\ 0 \le \sum_{i=1}^{p} z_{i11} \le z, \\ &0 \le \sum_{i=1}^{p} z_{i01} \le \frac{1}{2} - z,\ 0 \le \sum_{i=1}^{p} z_{i10} \le \frac{1}{2} - z,\ 0 \le z \le \frac{1}{2}\}. \end{aligned}$$

Now we need some observations to simplify our calculations. By observation and considering the definition of orientation, we can obtain a new modulo $(4p+1)$-orientation by reversing the orientation of each edge in a modulo $(4p+1)$-orientation. This operation in fact swaps the in-degree and out-degree of each vertex and thus we have the following properties of function f defined above.

Lemma 2.6. *The function f has the following properties.*

1. *z_{i00} is symmetric to z_{i11}. Exactly, we have*

$$\begin{aligned} &f(z, z_{100}, \dots, z_{p00}, z_{111}, \dots, z_{p11}, z_{101}, \dots, z_{p01}, z_{110}, \dots, z_{p10}) \\ =\,&f(z, z_{111}, \dots, z_{p11}, z_{100}, \dots, z_{p00}, z_{101}, \dots, z_{p01}, z_{110}, \dots, z_{p10}). \end{aligned}$$

   Similarly, z_{i01} is symmetric to z_{i10}.

2. *Type "in" and type "out" in an orientation are symmetric. Exactly, we have*

$$\begin{aligned} &f(z, z_{100}, \dots, z_{p00}, z_{111}, \dots, z_{p11}, z_{101}, \dots, z_{p01}, z_{110}, \dots, z_{p10}) \\ =\,&f(\frac{1}{2} - z, z_{101}, \dots, z_{p01}, z_{110}, \dots, z_{p10}, z_{100}, \dots, z_{p00}, z_{111}, \dots, z_{p11}). \end{aligned}$$

This lemma follows from the expression of f directly. Moreover, we observe that if f reaches the maximum in the region R, then it must satisfy the following condition:

Lemma 2.7. *If f reaches the maximum value at*

$$(z, z_{100}, \dots, z_{p00}, z_{111}, \dots, z_{p11}, z_{101}, \dots, z_{p01}, z_{110}, \dots, z_{p10})$$

in the region R, then

$$z_{i00} = z_{i11} \ \text{ and } \ z_{i01} = z_{i10}$$

for $i = 1, \dots, p$.

Proof. Let $z_i = \frac{z_{i00}+z_{i11}}{2}$ for $i = 1, \dots, p$. Note that ϕ is convex, which implies that $\sum_{i=1}^{p}(\phi(z_{i00}) + \phi(z_{i11})) + \phi(z - \sum_{i=1}^{p} z_{i00}) + \phi(z - \sum_{i=1}^{p} z_{i11}) \ge 2\sum_{i=1}^{p} \phi(z_i) + 2\phi(z - \sum_{i=1}^{p} z_i)$, and thus

$$\begin{aligned} &f(z, z_1, \dots, z_p, z_1, \dots, z_p, z_{101}, \dots, z_{p01}, z_{110}, \dots, z_{p10}) \ge \\ &f(z, z_{100}, \dots, z_{p00}, z_{111}, \dots, z_{p11}, z_{101}, \dots, z_{p01}, z_{110}, \dots, z_{p10}). \end{aligned}$$

This is the same as z_{i01} and z_{i10}. □

Then we begin to analyze the partial derivatives of f. By Lemma 2.6, $\frac{\partial f}{\partial z_{i00}}$ and $\frac{\partial f}{\partial z_{i11}}$, $\frac{\partial f}{\partial z_{i01}}$ and $\frac{\partial f}{\partial z_{i10}}$ have similar expression, so we only calculate $\frac{\partial f}{\partial z}$, $\frac{\partial f}{\partial z_{i00}}$, $\frac{\partial f}{\partial z_{i01}}$ for simplicity.

$$\begin{aligned} \frac{\partial f}{\partial z} =&\ln(p + z + \sum_{i=1}^{p} i(z_{i00} + z_{i11} - z_{i01} - z_{i10})) \\ &- \ln(p + \frac{1}{2} - z - \sum_{i=1}^{p} i(z_{i00} + z_{i11} - z_{i01} - z_{i10})) \\ &- \ln(z - \sum_{i=1}^{p} z_{i00}) - \ln(z - \sum_{i=1}^{p} z_{i11}) \end{aligned}$$

$$+\ \ln(\frac{1}{2}-z-\sum_{i=1}^{p}z_{i01})+\ln(\frac{1}{2}-z-\sum_{i=1}^{p}z_{i10}).$$

$$\begin{aligned}\frac{\partial f}{\partial z_{i00}} =& \ln\binom{4p+1}{i}\binom{4p+1-i}{p-i}\binom{3p+1}{p-i}-\ln\binom{4p+1}{p}\binom{3p+1}{p}\\ &+\ i\ln(p+z+\sum_{i=1}^{p}i\,(z_{i00}+z_{i11}-z_{i01}-z_{i10}))\\ &-\ i\ln(p+\frac{1}{2}-z-\sum_{i=1}^{p}i(z_{i00}+z_{i11}-z_{i01}-z_{i10}))\\ &-\ \ln z_{i00}+\ln(z-\sum_{i=1}^{p}z_{i00}).\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial z_{i01}} =& \ln\binom{4p+1}{i}\binom{4p+1-i}{p-i}\binom{3p+1}{p-i}-\ln\binom{4p+1}{p}\binom{3p+1}{p}\\ &-\ i\ln(p+z+\sum_{i=1}^{p}i(z_{i00}+z_{i11}-z_{i01}-z_{i10}))\\ &+\ i\ln(p+\frac{1}{2}-z-\sum_{i=1}^{p}i(z_{i00}+z_{i11}-z_{i01}-z_{i10}))\\ &-\ \ln z_{i01}+\ln(\frac{1}{2}-z-\sum_{i=1}^{p}z_{i01}).\end{aligned}$$

Let $\varphi_i := \frac{\binom{p}{i}\binom{3p+1}{p-i}}{\binom{4p+1}{p}}$, $t := \frac{\binom{3p+1}{p}}{\binom{4p+1}{p}} = \varphi_0$, and we define a special point

$$\hat{\mathbf{z}} = (\hat{z}, \hat{z}_{100}, \ldots, \hat{z}_{p00}, \hat{z}_{111}, \ldots, \hat{z}_{p11}, \hat{z}_{101}, \ldots, \hat{z}_{p01}, \hat{z}_{110}, \ldots, \hat{z}_{p10}),$$

where $\hat{z} = \frac{1}{4}$, $\hat{z}_{i00} = \hat{z}_{i11} = \hat{z}_{i01} = \hat{z}_{i10} = \frac{\varphi_i}{4} = \frac{\binom{p}{i}\binom{3p+1}{p-i}}{4\binom{4p+1}{p}}$. Subsequently, we shall prove f reaches the maximum value at $\hat{\mathbf{z}}$ in the region R. Before that, we show the unique maximum point of exponential term $f(\mathbf{z})$ actually determines the maximum value of $\mathbb{E}[Y^2]$.

Lemma 2.8. *When $p = 2, 3, 4$, if $\hat{\mathbf{z}}$ is the unique maximum point of $f(\mathbf{z})$ in the region R, then*

$$\mathbb{E}[Y^2] \sim \frac{(2\pi n)^{\frac{4p+1}{2}}}{\sqrt{|\det D^2 f(\hat{\mathbf{z}})|}} g(\hat{\mathbf{z}}) e^{f(\hat{\mathbf{z}})n}.$$

Proof. By Lemma 2.5, we have

$$\mathbb{E}[Y^2] \sim \sum_{\mathbf{z}\in L} S(\mathbf{z}) g(\mathbf{z}) e^{f(\mathbf{z})n},$$

where $S(\mathbf{z})$ is the error factor arising from the applications of Gosper's formula.

We denote the Hessian matrix of f evaluated at $\hat{\mathbf{z}}$ by $H := D^2 f(\hat{\mathbf{z}})$, and let $D := Df(\hat{\mathbf{z}})$ denote the gradient of f evaluated at $\hat{\mathbf{z}}$. We integrate near the maximum using a second-order Taylor series expansion. Let $[\mathbf{z}-\hat{\mathbf{z}}]$ denote a row vector, and let $[\mathbf{z}-\hat{\mathbf{z}}]^T$ be the transpose, a column vector. This gives that $f(\mathbf{z})$ near $\hat{\mathbf{z}}$ is equal to

$$\begin{aligned}f(\mathbf{z}) &= f(\hat{\mathbf{z}}) + D[\mathbf{z}-\hat{\mathbf{z}}]^T + \frac{1}{2}[\mathbf{z}-\hat{\mathbf{z}}]H[\mathbf{z}-\hat{\mathbf{z}}]^T + O(\|\mathbf{z}-\hat{\mathbf{z}}\|^3)\\ &= f(\hat{\mathbf{z}}) + \frac{1}{2}[\mathbf{z}-\hat{\mathbf{z}}]H[\mathbf{z}-\hat{\mathbf{z}}]^T + O(\|\mathbf{z}-\hat{\mathbf{z}}\|^3)\end{aligned}$$

by Taylor's Theorem. Please note that the error is valid provided that $\|\mathbf{z}-\hat{\mathbf{z}}\| = o(1)$.

Let $R' = \{\mathbf{z} \in R : \|\mathbf{z}-\hat{\mathbf{z}}\| \le n^{-2/5}\}$ and $L' = L \cap R'$. Since $S(\mathbf{z}) \sim 1$ and $\|\mathbf{z}-\hat{\mathbf{z}}\|^3 \le n^{-6/5}$, we have for all $\mathbf{z} \in R'$, $S(\mathbf{z})g(\mathbf{z}) \sim g(\hat{\mathbf{z}})$. Thus we have

$$\sum_{\mathbf{z}\in L'} S(\mathbf{z}) \frac{g(\mathbf{z})}{n} e^{f(\mathbf{z})n} \sim \frac{g(\hat{\mathbf{z}})}{n} \sum_{\mathbf{z}\in L'} e^{f(\mathbf{z})n}$$

$$\sim \frac{g(\hat{\mathbf{z}})}{n} e^{f(\hat{\mathbf{z}})n} \sum_{\mathbf{z}\in L'} \exp\left(\frac{1}{2}[\mathbf{z}-\hat{\mathbf{z}}]H[\mathbf{z}-\hat{\mathbf{z}}]^T n\right) e^{nO(\|\mathbf{z}-\hat{\mathbf{z}}\|^3)}$$
$$\sim \frac{g(\hat{\mathbf{z}})}{n} e^{f(\hat{\mathbf{z}})n} \sum_{\mathbf{z}\in L'} \exp\left(\frac{1}{2}[\mathbf{z}-\hat{\mathbf{z}}]H[\mathbf{z}-\hat{\mathbf{z}}]^T n\right),$$

where the last part follows since $e^{n\cdot O(\|\mathbf{z}-\hat{\mathbf{z}}\|^3)} = e^{n\cdot O(n^{-6/5})} = e^{O(n^{-1/5})}$ goes to 1 as n goes to infinity.

If we divide the sum by a factor of n^{4p+1}, then this becomes a Riemann sum over R'. Riemann sum in turn approximates an integral as $n \to \infty$. Hence

$$\sum_{\mathbf{z}\in L'} \exp\left(\frac{1}{2}[\mathbf{z}-\hat{\mathbf{z}}]H[\mathbf{z}-\hat{\mathbf{z}}]^T n\right) \sim n^{4p+1} \int_{\mathbf{z}\in R'} \exp\left(\frac{1}{2}[\mathbf{z}-\hat{\mathbf{z}}]H[\mathbf{z}-\hat{\mathbf{z}}]^T n\right) \mathrm{d}\mathbf{z}.$$

Now we change variables by letting $\mathbf{w} = \sqrt{n}(\mathbf{z}-\hat{\mathbf{z}})$ for $\mathbf{z}$. Note that the region of $\mathbf{w}$ corresponding to R' approaches the whole real plane since the original side length of the box R' is $n^{-2/5}$ and hence the scaled region has side length $n^{-2/5}\sqrt{n} = n^{1/10}$ which goes to infinity as n goes to infinity. Thus, this change of variable transforms the integral into

$$\int_{\mathbf{z}\in R'} \exp\left(\frac{1}{2}[\mathbf{z}-\hat{\mathbf{z}}]H[\mathbf{z}-\hat{\mathbf{z}}]^T n\right) \mathrm{d}\mathbf{z} \sim \frac{1}{n^{\frac{4p+1}{2}}} \int_{R^{4p+1}} \exp\left(\mathbf{w}\frac{H}{2}\mathbf{w}^T\right) \mathrm{d}\mathbf{w},$$

where H is the Hessian matrix of f evaluated at $\hat{\mathbf{z}}$. Diagonalizing and using the Gaussian integral, that is $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$, we see that the integral evaluates to

$$\frac{1}{n^{\frac{4p+1}{2}}}\sqrt{\frac{\pi^{4p+1}}{|\det \frac{H}{2}|}} = \frac{1}{n^{\frac{4p+1}{2}}}\sqrt{\frac{(2\pi)^{4p+1}}{|\det H|}} = \left(\frac{2\pi}{n}\right)^{\frac{4p+1}{2}} \frac{1}{\sqrt{|\det H|}}.$$

Therefore,

$$\sum_{\mathbf{z}\in L'} S(\mathbf{z})\frac{g(\mathbf{z})}{n} e^{f(\mathbf{z})n} \sim \frac{(2\pi n)^{\frac{4p+1}{2}}}{\sqrt{|\det H|}} g(\hat{\mathbf{z}}) e^{f(\hat{\mathbf{z}})n}.$$

Since H is negative definite, the value of f on the boundary of R' is $f(\hat{\mathbf{z}}) - \Omega(n^{-\frac{4}{5}})$. However, f is independent of n and $\hat{\mathbf{z}}$ is a global maximum. Thus,

$$\max_{\mathbf{z}\in R\setminus R'} f(\mathbf{z}) = f(\hat{\mathbf{z}}) - \Omega(n^{-\frac{4}{5}}).$$

Observe that

$$E[Y^2] = \sum_{\mathbf{z}\in L} S(\mathbf{z})g(\mathbf{z})e^{f(\mathbf{z})n}$$
$$= \sum_{\mathbf{z}\in L\setminus L'} S(\mathbf{z})g(\mathbf{z})e^{f(\mathbf{z})n} + \sum_{\mathbf{z}\in L'} S(\mathbf{z})g(\mathbf{z})e^{f(\mathbf{z})n}$$
$$\sim \sum_{\mathbf{z}\in L\setminus L'} S(\mathbf{z})g(\mathbf{z})e^{f(\mathbf{z})n} + \frac{(2\pi n)^{\frac{4p+1}{2}}}{\sqrt{|\det H|}} g(\hat{\mathbf{z}})e^{f(\hat{\mathbf{z}})n}.$$

Now consider $\mathbf{z} \in L\setminus L'$. Since $L\setminus L' \subset R\setminus R'$, we have that

$$e^{f(\mathbf{z})n} = e^{f(\hat{\mathbf{z}})n} \exp(-\Omega(n^{1/5})).$$

We have that $S(\mathbf{z}) = O(1)$ and $g(\mathbf{z}) = O(n^{4p+1})$ as each of the terms in the denominator of g is $O(n^{4p+1})$. Thus for each $\mathbf{z} \in L\setminus L'$, we see that

$$S(\mathbf{z})g(\mathbf{z})e^{f(\mathbf{z})n} = e^{f(\hat{\mathbf{z}})n} \exp(-\Omega(n^{1/5})).$$

The number of points in $L\setminus L'$ is polynomial, namely there are at most n^{4p+1}, the sum over points in $L\setminus L'$ is also $e^{f(\hat{\mathbf{z}})n}\exp(-\Omega(n^{1/5}))$. Therefore, $\mathbb{E}[Y^2] \sim \frac{(2\pi n)^{\frac{4p+1}{2}}}{\sqrt{|\det H|}} g(\hat{\mathbf{z}})e^{f(\hat{\mathbf{z}})n}$, as desired. □

Substitute $\hat{\mathbf{z}}$ into $f(\mathbf{z})$ and $g(\mathbf{z})$, we have

$$e^{f(\hat{\mathbf{z}})n} = \left(\frac{\binom{4p+1}{p}^2}{2^{4p-1}}\right)^n,$$

$$g(\hat{\mathbf{z}})(2\pi n)^{\frac{4p+1}{2}} = \frac{4(4p+1)}{\sqrt{2}t^2} \prod_{i=1}^{p} \frac{16}{\varphi_i{}^2}.$$

Recall that $\mathbb{E}[Y] \sim \sqrt{4p+1}\left(\frac{\sqrt{2}\binom{4p+1}{p}}{2^{2p}}\right)^n$. If $\hat{\mathbf{z}}$ is the global maximum point of f in the region R, then by Lemmas 2.5 and 2.8 we have

$$\frac{\mathbb{E}[Y^2]}{(\mathbb{E}[Y])^2} \sim \frac{2^{4p+3/2}}{t^2\sqrt{|\det H|}}\prod_{i=1}^{p}\frac{1}{\varphi_i{}^2}.$$

Together with $\exp\left(\sum_i \lambda_i\delta_i^2\right) = \frac{4p+1}{\sqrt{(4p+1)^2-4p^3}}$, it suffices to verify that

$$\hat{\mathbf{z}} \text{ is the unique global maximum point of } f \text{ in the region } R, \tag{$*$}$$

and

$$|\det H| = \frac{2^{8p+3}((4p+1)^2-4p^3)}{t^4(4p+1)^2}\prod_{i=1}^{p}\frac{1}{\varphi_i{}^4} = \frac{2^{8p+3}((4p+1)^2-4p^3)}{(4p+1)^2}\prod_{i=0}^{p}\frac{1}{\varphi_i{}^4} \tag{$**$}$$

to show that the condition (4) of Theorem 1.4 holds and thus complete our proof of Theorem 1.3. We verify $(**)$ by calculating the determinant of Hessian matrix in Appendix B, and the following theorem verifies $(*)$.

Theorem 2.9. *When $p = 2, 3, 4$, $\hat{\mathbf{z}}$ is the unique global maximal point of f within the region R.*

This result can be proved by showing that there is no maximum on the boundary and $\hat{\mathbf{z}}$ is the only possible maximal stationary point in the interior as in the following two Sections 2.3.1 and 2.3.2.

2.3.1. No maximum on boundary

Recall that

$$\begin{aligned} R = \{&(z, z_{100}, \dots, z_{p00}, z_{111}, \dots, z_{p11}, z_{101}, \dots, z_{p01}, z_{110}, \dots, z_{p10}) : \\ &z,\ z_{i00},\ z_{i01},\ z_{i10}\, z_{i11} \in \mathbb{R}^+,\ 0 \le \sum_{i=1}^{p} z_{i00} \le z,\ 0 \le \sum_{i=1}^{p} z_{i11} \le z, \\ &0 \le \sum_{i=1}^{p} z_{i01} \le \frac{1}{2} - z,\ 0 \le \sum_{i=1}^{p} z_{i10} \le \frac{1}{2} - z,\ 0 \le z \le \frac{1}{2}\}. \end{aligned}$$

Notice that if $\mathbf{z}$ is a point on the boundary of R, then any neighborhood of $\mathbf{z}$ must intersect with $\mathbb{R}^{4p+1}\backslash R$. Now we show that f cannot reach its maximum on boundary.

1. When $z \in (0, \frac{1}{2})$, if $\sum_{i=1}^{p} z_{i00} = z$, then there exist i_0 such that $z_{i_000} \neq 0$, and $\frac{\partial f}{\partial z_{i_000}} \to -\infty$ as z_{i_000} approaches to the boundary $\sum_{i=1}^{p} z_{i00} = z$, which indicates that such $\mathbf{z}$ cannot be a maximum point.
2. When $z \in (0, \frac{1}{2})$, if $\sum_{i=1}^{p} z_{i00} < z$ and there exist i_0 such that $z_{i_000} = 0$, then $\frac{\partial f}{\partial z_{i_000}} \to +\infty$ as $z_{i_000} \to 0^+$, which indicates that such $\mathbf{z}$ cannot be a maximum point.
3. When $z \in (0, \frac{1}{2})$, if $\sum_{i=1}^{p} z_{i01} = \frac{1}{2} - z$, then there exist i_0 such that $z_{i_001} \neq 0$, and $\frac{\partial f}{\partial z_{i_001}} \to -\infty$ as z_{i_001} approaches to the boundary $\sum_{i=1}^{p} z_{i01} = \frac{1}{2} - z$, which indicates that such $\mathbf{z}$ cannot be a maximum point.
4. When $z \in (0, \frac{1}{2})$, if $\sum_{i=1}^{p} z_{i01} < \frac{1}{2} - z$ and there exist i_0 such that $z_{i_001} = 0$, then $\frac{\partial f}{\partial z_{i_001}} \to +\infty$ as $z_{i_001} \to 0^+$, which indicates that such $\mathbf{z}$ cannot be a maximum point.
5. When $z = 0$, if $\sum_{i=1}^{p} z_{i01} < \frac{1}{2}$, then $\frac{\partial f}{\partial z} \to +\infty$ as $z \to 0^+$, which indicates that such $\mathbf{z}$ cannot be a maximum point.
6. When $z = \frac{1}{2}$, if $\sum_{i=1}^{p} z_{i00} < \frac{1}{2}$, then $\frac{\partial f}{\partial z} \to -\infty$ as $z \to \frac{1}{2}$, which indicates that such $\mathbf{z}$ cannot be a maximum point.
7. When $z = 0$, if $\sum_{i=1}^{p} z_{i01} = \frac{1}{2}$ and $z_{p01} < \frac{1}{2}$, then there exists i_0 such that $z_{i_001} > 0$, $\frac{\partial f}{\partial z_{i_001}} \to -\infty$ as z_{i_001} approaches to the boundary $\sum_{i=1}^{p} z_{i01} = \frac{1}{2}$, which indicates that such $\mathbf{z}$ cannot be a maximum point.
8. When $z = \frac{1}{2}$, if $\sum_{i=1}^{p} z_{i00} = \frac{1}{2}$ and $z_{p00} < \frac{1}{2}$, then there exists i_0 such that $z_{i_000} > 0$, $\frac{\partial f}{\partial z_{i_000}} \to -\infty$ as z_{i_000} approaches to the boundary $\sum_{i=1}^{p} z_{i00} = \frac{1}{2}$, which indicates that such $\mathbf{z}$ cannot be a maximum point.
9. When $z = 0$, if $\sum_{i=1}^{p} z_{i01} = \frac{1}{2}$ and $z_{p01} = \frac{1}{2}$, which indicates that $\mathbf{z} = (0, 0, \dots, 0, \dots, 0, \dots, 0, \frac{1}{2}, 0, \dots, 0, \frac{1}{2})$, $f(\mathbf{z}) = -\frac{4p+1}{2}\ln(4p+1) + \ln\binom{4p+1}{p} + \phi(\frac{4p+1}{2}) - 2\phi(\frac{1}{2}) = \ln\binom{4p+1}{p} - \frac{4p-1}{2}\ln 2 = \frac{1}{2}f(\hat{z})$.
10. When $z = \frac{1}{2}$, if $\sum_{i=1}^{p} z_{i00} = \frac{1}{2}$ and $z_{p00} = \frac{1}{2}$, which indicates that $\mathbf{z} = (\frac{1}{2}, 0, \dots, 0, \frac{1}{2}, 0, \dots, 0, \frac{1}{2}, 0, \dots, 0, 0, \dots, 0)$, $f(\mathbf{z}) = -\frac{4p+1}{2}\ln(4p+1) + \ln\binom{4p+1}{p} + \phi(\frac{4p+1}{2}) - 2\phi(\frac{1}{2}) = \ln\binom{4p+1}{p} - \frac{4p-1}{2}\ln 2 = \frac{1}{2}f(\hat{z})$.

Thus f obtains its maximum in the interior of the region R.

2.3.2. Unique maximum in the interior

In this part, we shall show that $\hat{\mathbf{z}}$ is the only stationary point in the interior. Let $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z_{i11}} = \frac{\partial f}{\partial z_{i00}} = \frac{\partial f}{\partial z_{i01}} = \frac{\partial f}{\partial z_{i10}} = 0$, and

$$b := p + z + \sum_{i=1}^{p} i(z_{i00} + z_{i11} - z_{i10} - z_{i01}), \tag{3}$$

$$x := \frac{z - \sum_{i=1}^{p} z_{i00}}{\frac{1}{2} - z - \sum_{i=1}^{p} z_{i01}},\ \alpha := z - \sum_{i=1}^{p} z_{i00}, \tag{4}$$

$$f(x) := \sum_{i=1}^{p} \frac{\varphi_i}{t} x^{2i},$$

$$z_{i00} = z_{i11},\ z_{i01} = z_{i10}.$$

Then we get

$$\frac{b}{2p + \frac{1}{2} - b} = \left(\frac{z - \sum_{i=1}^{p} z_{i00}}{\frac{1}{2} - z - \sum_{i=1}^{p} z_{i01}} \right)^2 = x^2, \tag{5}$$

$$z_{i00} = z_{i11} = \frac{\varphi_i}{t} \left(\frac{b}{2p + \frac{1}{2} - b} \right)^i \left(z - \sum_{i=1}^{p} z_{i00} \right) = \frac{\varphi_i}{t} x^{2i} \alpha, \tag{6}$$

$$z_{i01} = z_{i10} = \frac{\varphi_i}{t} \left(\frac{b}{2p + \frac{1}{2} - b} \right)^{-i} \left(\frac{1}{2} - z - \sum_{i=1}^{p} z_{i01} \right) = \frac{\varphi_i}{t} \frac{\alpha}{x} x^{-2i}, \tag{7}$$

and we derive the following observations:

Observation 2.10. *The equations $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z_{i11}} = \frac{\partial f}{\partial z_{i00}} = \frac{\partial f}{\partial z_{i01}} = \frac{\partial f}{\partial z_{i10}} = 0$ have a solution $x = 1$, and if $x > 0$ is a solution, then $\frac{1}{x}$ is also a solution.*

Observation 2.11. *Suppose $x \in (0, 1)$, i, j, s, t are positive integers with $i < j$ and $s < t$. Then $x^i(1 - x^t) > x^j(1 - x^s)$.*

Now we are ready to show $\hat{\mathbf{z}}$ is the only stationary point in the interior.

Lemma 2.12. *When $p = 2, 3, 4$, $x = 1$ is the unique positive solution of $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z_{i11}} = \frac{\partial f}{\partial z_{i00}} = \frac{\partial f}{\partial z_{i01}} = \frac{\partial f}{\partial z_{i10}} = 0$.*

Proof. By Observation 2.10, it suffices to prove that there is no solution in the interval $(0, 1)$. Sum up (6) and (7) over $i = 1, \ldots, p$, we obtain

$$\sum_{i=1}^{p} z_{i00} = \alpha f(x) = z - \alpha,$$

$$\sum_{i=1}^{p} z_{i01} = \frac{\alpha}{x} f(x^{-1}) = \frac{1}{2} - z - \frac{\alpha}{x},$$

which indicates that

$$z = \alpha(f(x) + 1).$$

Substituting above equations into (4), we have

$$x = \frac{\alpha}{\frac{1}{2} - \alpha(f(x) + 1) - \frac{\alpha}{x} f(x^{-1})},$$

$$\alpha = \frac{x}{2(1 + x(f(x) + 1) + f(x^{-1}))}.$$

Then we obtain

$$z = \frac{x(f(x) + 1)}{2(1 + x(f(x) + 1) + f(x^{-1}))},$$

$$z_{i00} = z_{i11} = \frac{\varphi_i}{t} \frac{x^{2i+1}}{2(1 + x(f(x) + 1) + f(x^{-1}))},$$

$$z_{i01} = z_{i10} = \frac{\varphi_i}{t}\frac{x^{-2i}}{2(1 + x(f(x) + 1) + f(x^{-1}))}.$$

Substituting α, z and x into (3), we have

$$b = p + \frac{x(f(x) + 1)}{2(1 + x(f(x) + 1) + f(x^{-1}))} + \sum_{i=1}^{p} \frac{2i\varphi_i(x^{2i+1} - x^{-2i})}{2t(1 + x(f(x) + 1) + f(x^{-1}))}.$$

Notice that (5) is equivalent to

$$(x^2 + 1)b = \left(2p + \frac{1}{2}\right) x^2.$$

Above two equations together give

$$\begin{aligned}(x^2 + 1)&\left(2pt(1 + x(f(x) + 1) + f(x^{-1})) + tx(f(x) + 1) + \sum_{i=1}^{p} 2i\varphi_i(x^{2i+1} - x^{-2i})\right)\\ &= \left(2p + \frac{1}{2}\right) 2t\left(1 + x(f(x) + 1) + f(x^{-1})\right) x^2,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\left(2px^2 - (2p + 1)\right) x(f(x) + 1) + \left((2p + 1)x^2 - 2p\right)\left(1 + f(x^{-1})\right)\\ &= (x^2 + 1)\sum_{i=1}^{p} \frac{2i\varphi_i(x^{2i+1} - x^{-2i})}{t}.\end{aligned}$$

Substituting the expression of $f(x)$ into above equation and multiplying by x^{2p}, we have

$$\begin{aligned}&\left(2px^2 - (2p + 1)\right)\sum_{i=0}^{p} \frac{\varphi_i}{t}x^{2p+2i+1} + \left((2p + 1)x^2 - 2p\right)\sum_{i=0}^{p} \frac{\varphi_i}{t}x^{2p-2i}\\ &= (x^2 + 1)\sum_{i=1}^{p} \frac{2i\varphi_i(x^{2p+2i+1} - x^{2p-2i})}{t},\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\sum_{i=0}^{p} \varphi_i\left((2p - 2i)x^2 - (2p + 2i + 1)\right) x^{2p+2i+1} +\\ &\qquad \sum_{i=0}^{p} \varphi_i\left((2p + 2i + 1)x^2 - (2p - 2i)\right) x^{2p-2i} = 0.\end{aligned}$$

Then utilize the symmetry of the two polynomials to simplify the equation:

$$\begin{aligned}&\sum_{i=1}^{p} \left(\varphi_{i-1}(2p - 2i + 2) - \varphi_i(2p + 2i + 1)\right) x^{2p+2i+1} - \varphi_0(2p + 1)x^{2p+1} +\\ &\qquad \sum_{i=0}^{p-1} \left(\varphi_{i+1}(2p + 2i + 3) - \varphi_i(2p - 2i)\right) x^{2p-2i} + \varphi_0(2p + 1)x^{2p+2} = 0.\end{aligned}$$

Adjust the index and we obtain

$$\begin{aligned}&\sum_{i=1}^{p} \left(\varphi_{i-1}(2p - 2i + 2) - \varphi_i(2p + 2i + 1)\right) x^{2p+2i+1} - \varphi_0(2p + 1)x^{2p+1} +\\ &\qquad \sum_{i=1}^{p} \left(\varphi_i(2p + 2i + 1) - \varphi_{i-1}(2p - 2i + 2)\right) x^{2p-2i+2} + \varphi_0(2p + 1)x^{2p+2} = 0.\end{aligned}$$

Let $\varphi_{-1} = 0$. Then we have

$$\sum_{i=0}^{p} \left(\varphi_{i-1}(2p - 2i + 2) - \varphi_i(2p + 2i + 1)\right)(x^{2p+2i+1} - x^{2p-2i+2}) = 0.$$

1. When $p=2$:

$$\left[2\binom{2}{1}\binom{7}{1}-9\right](x^9-x^2)+\left[4\binom{7}{2}-7\binom{2}{1}\binom{7}{1}\right](x^7-x^4)-5\binom{7}{2}(x^5-x^6)=0,$$

$$19(x^9-x^2)-15(x^7-x^4)-105(x^5-x^6)=0,$$

which is equivalent to

$$15\left[x^2(x^7-1)-x^4(x^3-1)\right]+4(x^9-x^2)+105(x^6-x^5)=0.$$

By Observation 2.11, the left of above equation is negative when $x\in(0,1)$, thus we complete the proof when $p=2$.

2. When $p=3$:

$$\left[2\binom{3}{2}\binom{10}{1}-13\right](x^{13}-x^2)+\left[4\binom{3}{1}\binom{10}{2}-11\binom{3}{2}\binom{10}{1}\right](x^{11}-x^4)$$
$$+\left[6\binom{10}{3}-9\binom{3}{1}\binom{10}{2}\right](x^9-x^6)-7\binom{10}{3}(x^7-x^8)=0,$$

which is equivalent to

$$47(x^{13}-x^2)+210(x^{11}-x^4)-495(x^9-x^6)-840(x^7-x^8)=0.$$

Notice that $-495(x^9-x^6)$ is the only positive term. When $x<\frac{1}{2}$, one can check that the left of above equation is negative, thus we can assume that $x\geq\frac{1}{2}$. Consider the following decomposition of coefficient:

$$\left[47(x^{13}-x^2)+210(x^{11}-x^4)-257(x^9-x^6)\right]-x^6\left[238(x^3-1)+840(x-x^2)\right]=0,$$

which is equivalent to

$$\left[47(x^{13}-x^2)+210(x^{11}-x^4)-257(x^9-x^6)\right]-x^6(1-x)(-238x^2+602x-238)=0.$$

Now the left hand side of above equation is negative when $x\in[\frac{1}{2},1)$, thus we complete the proof when $p=3$.

3. When $p=4$:

$$\left[2\binom{4}{3}\binom{13}{1}-17\right](x^{17}-x^2)+\left[4\binom{4}{2}\binom{13}{2}-15\binom{4}{3}\binom{13}{1}\right](x^{15}-x^4)$$
$$+\left[6\binom{4}{1}\binom{13}{3}-13\binom{4}{2}\binom{13}{2}\right](x^{13}-x^6)+\left[8\binom{13}{4}-11\binom{4}{1}\binom{13}{3}\right](x^{11}-x^8)$$
$$-9\binom{13}{4}(x^9-x^{10})=0,$$

which is equivalent to

$$87x^2(x^{15}-1)+1092x^4(x^{11}-1)+780x^6(x^7-1)-6864x^8(x^3-1)-6435x^9(1-x)=0.$$

Observe that $-6864x^8(x^3-1)$ is the unique positive term and we can further obtain

$$\frac{87}{x^6}\frac{x^{15}-1}{x^3-1}+\frac{1092}{x^4}\frac{x^{11}-1}{x^3-1}+\frac{780}{x^2}\frac{x^7-1}{x^3-1}+6435x\frac{x-1}{x^3-1}=6864. \tag{8}$$

Since $\frac{x^{15}-1}{x^3-1},\frac{x^{11}-1}{x^3-1},\frac{x^7-1}{x^3-1}\geq 1$, one can find that the left hand side of above equation is larger that 6864 when $x\in(0,0.7]$. Note that if $q>p$, then

$$\frac{\partial}{\partial x}\left(\frac{x^q-1}{x^p-1}\right)=\frac{x^{p-1}((q-p)x^q-qx^{q-p}+p)}{(x^p-1)^2},$$

$$\frac{\partial}{\partial x}((q-p)x^q-qx^{q-p}+p)=q(q-p)x^{q-1}(1-x^{-p})\leq 0,$$

which indicates that $\frac{x^q-1}{x^p-1}$ is increasing in $(0,1)$ since the numerator of $\frac{\partial}{\partial x}\left(\frac{x^q-1}{x^p-1}\right)$ is non-negative. Similarly, $\frac{x^p-1}{x^q-1}$ is decreasing in $(0,1)$. Thus when $x\in(0.7,0.8]$, $\frac{x^q-1}{x^p-1}$ is bounded below by taking $x=0.7$ and $\frac{x^p-1}{x^q-1}$ is bounded below by taking $x=0.8$, and the left hand side of (8) is at least

$$\begin{aligned}&\frac{87}{x^6}\times 1.514+\frac{1092}{x^4}\times 1.49+\frac{780}{x^2}\times 1.396+6435x\times 0.409\\ \geq\ &\frac{87}{0.8^6}\times 1.514+\frac{1092}{0.8^4}\times 1.49+\frac{780}{0.8^2}\times 1.396+6435\times 0.7\times 0.409\\ \geq\ &7000,\end{aligned}$$

which indicates that there is no solution in $(0.7, 0.8]$. This argument can show that (8) has no solution in $[0.8, 0.9]$, $[0.9, 0.93]$ and $[0.93, 1)$ as well.

Thus we complete the proof of Lemma 2.12. □

At this point, based on Sections 2.3.1 and 2.3.2, we have proved Theorem 2.9 and verified condition $(*)$. Moreover, condition $(**)$ has been confirmed in Appendix B. According to the discussion preceding Theorem 2.9, this completes the proof of Theorem 1.3.

3. Concluding remarks

By above calculation, we have the following corollary.

Corollary 3.1. *For $p \geq 5$, if $\hat{\mathbf{z}}$ is the only maximum of f in region R, then*

$$\frac{\mathbb{E}[Y^2]}{(\mathbb{E}[Y])^2} \sim \frac{4p+1}{\sqrt{4p^3-(4p+1)^2}}.$$

When $p = 5$, there is a positive eigenvalue of the Hessian matrix of f at $\hat{\mathbf{z}}$, which indicates that the global maximum cannot be obtained at $x = 1$ and $\frac{\mathbb{E}[Y^2]}{(\mathbb{E}[Y])^2} \to \infty$.

As introduced in the first section, Alon and Prałat [1] proved that there exists large enough p_0 such that for all $p > p_0$, almost all $(4p+1)$-regular graphs have modulo $(2p+1)$-orientations. By rough calculation, one would find that 10^{13} is sufficient. The value of p between 5 and 10^{13} remains as an open problem and requires new tools.

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Appendix A. Details of verifying condition (2)

Assume that $j_1, \ldots, j_m$ are non-negative integers. Now we verify

$$\frac{\mathbb{E}[Y[X_1]_{j_1}\ldots[X_m]_{j_m}]}{\mathbb{E}Y} \to \prod_{i=1}^{m}(\lambda_i(1+\delta_i))^{j_i},$$

where $\lambda_i = \frac{(d-1)^j}{2j}$ and $\delta_i = (-\frac{p}{4p+1})^i$. Since $\mathbb{E}[Y[X_1]_{j_1}\ldots[X_m]_{j_m}]$ and $\mathbb{E}[Y]$ are in the same probability space, we only need to divide the total number of $Y[X_1]_{j_1}\ldots[X_m]_{j_m}$ by the total number of Y. We denote the condition

all the cycles of length between 1 and m are pairwise disjoint (CDJ)

by (CDJ). Notice that $[X_k]_{j_k}$ represents the number of tuples of different k-cycles, to calculate the total number of $Y[X_1]_{j_1}\ldots[X_m]_{j_m}$, we only need to consider the following factors:

1. The first factor we consider is the number of all possible cycles of desired length in an n-vertex graph. Note that the number of considered C_i is exactly j_i for $1 \leq i \leq m$. Assume that these cycles edge-induce a graph of j vertices and have h edges in total. After enumerating these factors, we demonstrate that the summation over pairwise disjoint cycles determines the total number of $Y[X_1]_{j_1}\ldots[X_m]_{j_m}$.
2. Let $l \in \{1, \ldots, j_k\}$ and $k \in \{1, \ldots, m\}$. If (CDJ) holds, then the number of orientations of lth cycle of length k with exactly s_{kl} sinks and s_{kl} sources is $2\binom{k}{2s_{kl}}$. Otherwise, such amount is bounded by a constant independent of n.
3. If (CDJ) holds, for s_{kl} sinks in the cycle mentioned above, we can suppose that there are exactly x_{1kl} out-vertex and $(s_{kl} - x_{1kl})$ in-vertex among them. For s_{kl} sources, suppose that there are exactly x_{2kl} out-vertex and $(s_{kl} - x_{2kl})$ in-vertex among them. For the other $(k - 2s_{kl})$ vertices in the cycle, suppose that there are x_{3kl} out-vertex and $(k - 2s_{kl} - x_{3kl})$ in-vertex.
4. If (CDJ) holds, for $(n-j)$ vertices other than these cycles, there are

$$\frac{n}{2} - \sum_{k=1}^{m}\sum_{l=1}^{j_k}\sum_{g=1}^{3} x_{gkl}$$

out-vertices and

$$\frac{n}{2}+\sum_{k=1}^{m}\sum_{l=1}^{j_k}\sum_{g=1}^{3}x_{gkl}-j$$

in-vertices.

5. After partitioning vertices into in-vertex set and out-vertex set, we then determine the in-points and out-points in each cell, and match them. The process of matching gives a factor $(\frac{(4p+1)n}{2}-h)!$.
6. Finally, we divide the product of these factors by the number of all modulo $(2p+1)$-orientations in $(4p+1)$-regular pairings.

Recall that

$$\begin{aligned}
&\mathbb{E}[YX_j]\cdot M((4p+1)n)\\
&=\frac{1}{2}\binom{n}{j}(j-1)!(4p(4p+1))^j\\
&\quad\sum_{s=0}^{\lfloor\frac{j}{2}\rfloor}2\binom{j}{2s}\sum_{x_1=0}^{s}\binom{s}{x_1}\binom{4p-1}{p}^{x_1}\binom{4p-1}{p-2}^{s-x_1}\sum_{x_2=0}^{s}\binom{s}{x_2}\binom{4p-1}{p-2}^{x_2}\binom{4p-1}{p}^{s-x_2}\\
&\quad\sum_{x_3=0}^{j-2s}\binom{j-2s}{x_3}\binom{4p-1}{p-1}^{j-2s}\binom{n-j}{\frac{n}{2}-x_1-x_2-x_3}\binom{4p+1}{p}^{n-j}\left(\frac{(4p+1)n}{2}-j\right)!\\
&=C_{p,j}\binom{n}{j}\binom{n}{\frac{n}{2}}\binom{4p+1}{p}^{n}\left(\frac{(4p+1)n}{2}-j\right)! \qquad \text{(A)}
\end{aligned}$$

where $C_{p,j}$ is a constant independent of n and the last equality follows by

$$\frac{\binom{n-j}{\frac{n}{2}-x_1-x_2-x_3}}{\binom{n}{\frac{n}{2}}}=\frac{(n-j)!}{n!}\frac{\frac{n}{2}!\frac{n}{2}!}{(\frac{n}{2}-x_1-x_2-x_3)!(\frac{n}{2}+x_1+x_2+x_3-j)!}\sim\frac{1}{2^j}.$$

Suppose that S is the total number of $Y[X_1]_{j_1}\dots[X_m]_{j_m}$, and S_1 is the total number of $Y[X_1]_{j_1}\dots[X_m]_{j_m}$ under the condition (CDJ), S_2 is the total number of $Y[X_1]_{j_1}\dots[X_m]_{j_m}$ violating the condition (CDJ). Then we have an expression of S_1 and S_2 similar to (A). That is,

$$S_1=C'_{p,j_0,m,j_i}\binom{n}{j_0}\binom{n}{\frac{n}{2}}\binom{4p+1}{p}^{n}\left(\frac{(4p+1)n}{2}-j_0\right)!,$$

and

$$S_2=\sum_{j=m}^{j_0-1}\sum_{h=j+1}^{j_0}C''_{p,j,h,m,j_i}\binom{n}{j}\binom{n}{\frac{n}{2}}\binom{4p+1}{p}^{n}\left(\frac{(4p+1)n}{2}-h\right)!,$$

where $j_0=\sum_{k=1}^{m}j_kk$.

Since $[X_k]_{j_k}$ represents the number of tuples of different k-cycles, all the considered cycles in S_2 together have more edges than vertices, and the range of h in the second summation starts from $j+1$. We shall show that $S_2=o(S_1)$.

Proposition A.1. *We have $S_2=o(S_1)$.*

Proof. We have

$$\begin{aligned}
S_2&=\sum_{j=m}^{j_0-1}\sum_{h=j+1}^{j_0}C''_{p,j,h,m,j_i}\binom{n}{j}\binom{4p+1}{p}^{n}\binom{n}{\frac{n}{2}}\left(\frac{(4p+1)n}{2}-h\right)!\\
&\le\sum_{j=m}^{j_0-1}j_0C''_{p,j,h,m,j_i}\binom{n}{j}\binom{4p+1}{p}^{n}\binom{n}{\frac{n}{2}}\left(\frac{(4p+1)n}{2}-j-1\right)!\\
&\le C_0\frac{1}{n}\binom{n}{j_0}\binom{4p+1}{p}^{n}\binom{n}{\frac{n}{2}}\left(\frac{(4p+1)n}{2}-j_0\right)!\\
&=o(S_1)
\end{aligned}$$

for some constant C_0. The last inequality follows from

$$\binom{n}{j}\left(\frac{(4p+1)n}{2}-j-1\right)! = \Theta\left(\frac{1}{n}\binom{n}{j_0}\left(\frac{(4p+1)n}{2}-j_0\right)!\right). \quad \square$$

The above proposition guarantees that

$$\frac{\mathbb{E}[Y[X_1]_{j_1}\dots[X_m]_{j_m}]}{\mathbb{E}Y} \sim \frac{S_1}{\mathbb{E}Y}.$$

In S_1, all the cycles we consider are pairwise disjoint. Recall that $j_0=\sum_{k=1}^m j_k k$, and let $c=j_1+\cdots+j_m$. Since

$$\begin{aligned}&\frac{\binom{n-j_0}{\frac{n}{2}-\sum_{k=1}^m\sum_{l=1}^{j_k}\sum_{g=1}^3 x_{gkl}}}{\binom{n}{n/2}}\\ &=\frac{(n-j_0)!}{n!}\frac{\frac{n}{2}!\frac{n}{2}!}{(\frac{n}{2}-(\sum_{k=1}^m\sum_{l=1}^{j_k}\sum_{g=1}^3 x_{gkl}))!(\frac{n}{2}+\sum_{k=1}^m\sum_{l=1}^{j_k}\sum_{g=1}^3 x_{gkl}-j_0)!}\sim\frac{1}{2^{j_0}}\end{aligned}$$

and take $\lambda_j=\frac{(4p)^j}{2j}$ and $\delta_j=(-\frac{p}{4p+1})^j$, we have

$$\begin{aligned}
\frac{S_1}{\mathbb{E}Y} &= \frac{1}{\binom{n}{n/2}\binom{4p+1}{p}^n\frac{(4p+1)n}{2}!}\frac{1}{2^c}\frac{n!}{\prod_{a=1}^m(a!)^{j_a}(n-j_0)!}\prod_{a=1}^m((a-1)!)^{j_a}(4p(4p+1))^{j_0}\\
&\prod_{k=1}^m\prod_{l=1}^{j_k}\sum_{s_{kl}=0}^{\lfloor\frac{k}{2}\rfloor}2\binom{k}{2s_{kl}}\sum_{x_{1kl}=0}^{s_{kl}}\binom{s_{kl}}{x_{1kl}}\binom{4p-1}{p}^{x_{1kl}}\binom{4p-1}{p-2}^{s_{kl}-x_{1kl}}\\
&\sum_{x_{2kl}=0}^{s_{kl}}\binom{s_{kl}}{x_{2kl}}\binom{4p-1}{p-2}^{x_{2kl}}\binom{4p-1}{p}^{s_{kl}-x_{2kl}}\sum_{x_{3kl}=0}^{k-2s_{kl}}\binom{k-2s_{kl}}{x_{3kl}}\binom{4p-1}{p-1}^{k-2s_{kl}}\\
&\binom{n-j_0}{\frac{n}{2}-\sum_{k=1}^m\sum_{l=1}^{j_k}\sum_{g=1}^3 x_{gkl}}\binom{4p+1}{p}^{n-j_0}\left(\frac{(4p+1)n}{2}-j_0\right)!\\
&\sim\frac{(4p(4p+1))^{j_0}n^{j_0}}{2^{j_0}\prod_{a=1}^m a^{j_a}\binom{4p+1}{p}^{j_0}(\frac{(4p+1)n}{2})^{j_0}}\\
&\prod_{k=1}^m\left(\sum_{s_{kl}=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2s_{kl}}\sum_{x_{1kl}=0}^{s_{kl}}\binom{s_{kl}}{x_{1kl}}\binom{4p-1}{p}^{x_{1kl}}\binom{4p-1}{p-2}^{s_{kl}-x_{1kl}}\right.\\
&\left.\sum_{x_{2kl}=0}^{s_{kl}}\binom{s_{kl}}{x_{2kl}}\binom{4p-1}{p-2}^{x_{2kl}}\binom{4p-1}{p}^{s_{kl}-x_{2kl}}\sum_{x_{3kl}=0}^{k-2s_{kl}}\binom{k-2s_{kl}}{x_{3kl}}\binom{4p-1}{p-1}^{k-2s_{kl}}\right)^{j_k}\\
&=\prod_{k=1}^m\left(\frac{(4p)^k}{k\binom{4p+1}{p}^k}\sum_{s_{kl}=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2s_{kl}}\left(\frac{\binom{4p-1}{p}+\binom{4p-1}{p-2}}{2\binom{4p-1}{p-1}}\right)^{2s_{kl}}2^k\binom{4p-1}{p-1}^k\right)^{j_k}\\
&=\prod_{k=1}^m\left(\frac{(4p)^k}{k}\sum_{s_{kl}=0}^{\lfloor\frac{k}{2}\rfloor}\binom{k}{2s_{kl}}\left(\frac{\binom{4p-1}{p}+\binom{4p-1}{p-2}}{2\binom{4p-1}{p-1}}\right)^{2s_{kl}}\left(\frac{2\binom{4p-1}{p-1}}{\binom{4p+1}{p}}\right)^k\right)^{j_k}\\
&=\prod_{k=1}^m(\lambda_k(1+\delta_k))^{j_k},
\end{aligned}$$

as desired. Note that the last equality holds by the proof of Lemma 2.3.

Appendix B. The Hessian matrix

In this appendix, we shall calculate the determinant of Hessian matrix of function f to verify (∗∗), and thus complete the whole proof of Theorem 1.3. Recall that $t = \frac{\binom{3p+1}{p}}{\binom{4p+1}{p}}$, $\varphi_i = \frac{\binom{p}{i}\binom{3p+1}{p-i}}{\binom{4p+1}{p}}$. By calculating the second-order partial derivative for $i, j \in \{1, \ldots, p\}$, we have

$$\frac{\partial^2 f}{\partial z^2}(\hat{\mathbf{z}}) = \frac{8}{4p+1} - \frac{16}{t}, \qquad \frac{\partial^2 f}{\partial z \partial z_{i00}}(\hat{\mathbf{z}}) = \frac{8i}{4p+1} + \frac{4}{t},$$

$$\frac{\partial^2 f}{\partial z \partial z_{i01}}(\hat{\mathbf{z}}) = -\frac{8i}{4p+1} - \frac{4}{t}, \qquad \frac{\partial^2 f}{\partial z_{i00}^2}(\hat{\mathbf{z}}) = \frac{8i^2}{4p+1} - \frac{4}{t} - \frac{4}{\varphi_i},$$

$$\frac{\partial^2 f}{\partial z_{i01} \partial z_{j10}}(\hat{\mathbf{z}}) = \frac{8ij}{4p+1}, \qquad \frac{\partial^2 f}{\partial z_{i00} \partial z_{j00}}(\hat{\mathbf{z}}) = \frac{8ij}{4p+1} - \frac{4}{t}, \quad i \neq j$$

$$\frac{\partial^2 f}{\partial z_{i00} \partial z_{j11}}(\hat{\mathbf{z}}) = \frac{8ij}{4p+1}, \qquad \frac{\partial^2 f}{\partial z_{i00} \partial z_{j01}}(\hat{\mathbf{z}}) = -\frac{8ij}{4p+1},$$

$$\frac{\partial^2 f}{\partial z_{i01}^2}(\hat{\mathbf{z}}) = \frac{8i^2}{4p+1} - \frac{4}{t} - \frac{4}{\varphi_i}, \qquad \frac{\partial^2 f}{\partial z_{i01} \partial z_{j01}}(\hat{\mathbf{z}}) = \frac{8ij}{4p+1} - \frac{4}{t}, \quad i \neq j.$$

Hence the Hessian matrix can be expressed as follows, and an explicit expression refers to Appendix C.

$$H = \begin{pmatrix} -\frac{20}{t} & & & & \\ & A & & & \\ & & A & & \\ & & & A & \\ & & & & A \end{pmatrix} - \frac{8}{t} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$+ \frac{8}{t} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} + \frac{4}{t} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$-\frac{4}{t} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix} + \frac{8}{4p+1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ p \\ 1 \\ \vdots \\ p \\ -1 \\ \vdots \\ -p \\ -1 \\ \vdots \\ -p \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & p & 1 & \cdots & p & -1 & \cdots & -p & -1 & \cdots & -p \end{pmatrix}$$

$$:= D - \frac{8}{t} v_1 v_1^T + \frac{8}{t} v_2 v_2^T + \frac{4}{t} v_3 v_3^T - \frac{4}{t} v_4 v_4^T + \frac{8}{4p+1} v_5 v_5^T,$$

where A is a matrix of order p and

$$A=\begin{pmatrix}-\frac{4}{t}-\frac{4}{\varphi_1} & -\frac{4}{t} & \cdots & -\frac{4}{t}\\ -\frac{4}{t} & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & -\frac{4}{t}\\ -\frac{4}{t} & \cdots & -\frac{4}{t} & -\frac{4}{t}-\frac{4}{\varphi_p}\end{pmatrix}=\begin{pmatrix}-\frac{4}{\varphi_1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & -\frac{4}{\varphi_p}\end{pmatrix}-\frac{4}{t}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\begin{pmatrix}1 & \cdots & 1\end{pmatrix}$$
$$:=D_0-\frac{4}{t}uu^T.$$

Then we shall calculate the determinant of H by the following equations (please refer to [17], equation (6.2.3) and (3.8.2)):

$$\det(B+uv^T)=(1+u^TB^{-1}v)\det B,\quad (B+uv^T)^{-1}=B^{-1}-\frac{B^{-1}uv^TB^{-1}}{1+u^TB^{-1}v}.$$

We have $1-\frac{4}{t}u^TD_0^{-1}u=\frac{1}{t}$, $\det A=\frac{(-1)^p}{t}\prod_{i=1}^{p}\frac{4}{\varphi_i}$ and

$$A^{-1}=\begin{pmatrix}-\frac{\varphi_1}{4} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & -\frac{\varphi_p}{4}\end{pmatrix}+\frac{1}{4}\begin{pmatrix}\varphi_1\\ \vdots\\ \varphi_p\end{pmatrix}\begin{pmatrix}\varphi_1 & \cdots & \varphi_p\end{pmatrix},$$

$$D^{-1}=\begin{pmatrix}-\frac{t}{20} & & & & \\ & A^{-1} & & & \\ & & A^{-1} & & \\ & & & A^{-1} & \\ & & & & A^{-1}\end{pmatrix}$$

and $\det D=-\frac{5}{2t}2^{8p+3}\prod_{i=0}^{p}\frac{1}{\varphi_i^4}$.

Let $H_1=D-\frac{8}{t}v_1v_1^T$, note that $-u^TA^{-1}u=\frac{t(t-1)}{4}$ and $A^{-1}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}=\begin{pmatrix}-\frac{t\varphi_1}{4}\\ \vdots\\ -\frac{t\varphi_p}{4}\end{pmatrix}$, thus we have $1-\frac{8}{t}v_1^TD^{-1}v_1=1+2(2-2t)=5-4t$, and

$$\begin{aligned}H_1^{-1}&=D^{-1}+\frac{8}{t(5-4t)}D^{-1}v_1v_1^TD^{-1}\\ &=D^{-1}+\frac{t}{2(5-4t)}w_1^Tw_1,\end{aligned}$$

where

$$w_1=(0,\varphi_1,\ldots,\varphi_p,\varphi_1,\ldots,\varphi_p,0,\ldots,0,0,\ldots,0).$$

Let $H_2=H_1+\frac{8}{t}v_2v_2^T$, we have $1+\frac{8}{t}v_2^TH_1^{-1}v_2=\frac{8t-5}{5(5-4t)}$, and

$$\begin{aligned}H_2^{-1}&=H_1^{-1}-\frac{40(5-4t)}{t(8t-5)}H_1^{-1}v_2v_2^TH_1^{-1}\\ &=H_1^{-1}-\frac{40(5-4t)}{t(8t-5)}w_2^Tw_2,\end{aligned}$$

where

$$w_2 = (-\tfrac{t}{20}, -\tfrac{t\varphi_1}{4(5-4t)}, \ldots, -\tfrac{t\varphi_p}{4(5-4t)}, -\tfrac{t\varphi_1}{4(5-4t)}, \ldots, -\tfrac{t\varphi_p}{4(5-4t)}, 0, \ldots, 0, 0, \ldots, 0).$$

Let $H_3 = H_2 + \frac{4}{t} v_3 v_3^T$. Then we have $1 + \frac{4}{t} v_3^T H_2^{-1} v_3 = \frac{16t^2-12t-1}{8t-5}$, and

$$\begin{aligned} H_3^{-1} &= H_2^{-1} - \frac{4(8t-5)}{t(16t^2-12t-1)} H_2^{-1} v_3 v_3^T H_2^{-1} \\ &= H_2^{-1} - \frac{4(8t-5)}{t(16t^2-12t-1)} w_3^T w_3, \end{aligned}$$

where w_3 is equal to

$$(-\tfrac{t(1-t)}{8t-5}, -\tfrac{3t\varphi_1}{4(8t-5)}, \ldots, -\tfrac{3t\varphi_p}{4(8t-5)}, -\tfrac{3t\varphi_1}{4(8t-5)}, \ldots, -\tfrac{3t\varphi_p}{4(8t-5)}, -\tfrac{\varphi_1 t}{4}, \ldots, -\tfrac{\varphi_p t}{4}, -\tfrac{\varphi_1 t}{4}, \ldots, -\tfrac{\varphi_p t}{4}).$$

Let $H_4 = H_3 - \frac{4}{t} v_4 v_4^T$. Then we have $1 - \frac{4}{t} v_4^T H_3^{-1} v_4 = \frac{4t}{16t^2-12t-1}$, and

$$\begin{aligned} H_4^{-1} &= H_3^{-1} + \frac{16t^2-12t-1}{t^2} H_3^{-1} v_4 v_4^T H_3^{-1} \\ &= H_3^{-1} + \frac{16t^2-12t-1}{t^2} w_4^T w_4, \end{aligned}$$

where w_4 is equal to

$$(-\tfrac{t}{4(16t^2-12t-1)}, -\varphi_1 \tfrac{t(4t+1)}{4(16t^2-12t-1)}, \ldots, -\varphi_p \tfrac{t(4t+1)}{4(16t^2-12t-1)}, -\varphi_1 \tfrac{t(4t+1)}{4(16t^2-12t-1)}, \ldots, -\varphi_p \tfrac{t(4t+1)}{4(16t^2-12t-1)},$$
$$-\varphi_1 \tfrac{t(4t-1)}{4(16t^2-12t-1)}, \ldots, -\varphi_p \tfrac{t(4t-1)}{4(16t^2-12t-1)}, -\varphi_1 \tfrac{t(4t-1)}{4(16t^2-12t-1)}, \ldots, -\varphi_p \tfrac{t(4t-1)}{4(16t^2-12t-1)}).$$

Now $H = H_4 + \frac{8}{4p+1} v_5 v_5^T$, $1 + \frac{8}{4p+1} v_5^T H_4^{-1} v_5 = \frac{(4p+1)^2-4p^3}{2(4p+1)^2}$.

$$\begin{aligned} \det H &= \left(1 + \frac{8}{4p+1} v_5^T H_4^{-1} v_5\right) \det H_4 \\ &= \cdots \\ &= \left(1 + \frac{8}{4p+1} v_5^T H_4^{-1} v_5\right)\left(1 - \frac{4}{t} v_4^T H_3^{-1} v_4\right)\left(1 + \frac{4}{t} v_3^T H_2^{-1} v_3\right) \\ &\qquad \left(1 + \frac{8}{t} v_2^T H_1^{-1} v_2\right)\left(1 - \frac{8}{t} v_1^T D^{-1} v_1\right) \det D. \end{aligned}$$

Recall that $\det D = -\frac{5}{2t} 2^{8p+3} \prod_{i=0}^{p} \frac{1}{\varphi_i^4}$, $1 - \frac{8}{t} v_1^T D^{-1} v_1 = 5 - 4t$, $1 + \frac{8}{t} v_2^T H_1^{-1} v_2 = \frac{8t-5}{5(5-4t)}$, $1 + \frac{4}{t} v_3^T H_2^{-1} v_3 = \frac{16t^2-12t-1}{8t-5}$, $1 - \frac{4}{t} v_4^T H_3^{-1} v_4 = \frac{4t}{16t^2-12t-1}$, $1 + \frac{8}{4p+1} v_5^T H_4^{-1} v_5 = \frac{(4p+1)^2-4p^3}{2(4p+1)^2}$. Hence

$$|\det H| = \frac{2^{8p+3}((4p+1)^2-4p^3)}{(4p+1)^2} \prod_{i=0}^{p} \frac{1}{{\varphi_i}^4},$$

which verifies (∗∗) and completes the proof of Theorem 1.3.

Appendix C. Explicit expression of Hessian matrix

The order of variables is

$$(z, z_{100}, \ldots, z_{p00}, z_{111}, \ldots, z_{p11}, z_{101}, \ldots, z_{p01}, z_{110}, \ldots, z_{p10}).$$

Now $H-\frac{8}{4p+1}v_5v_5^T$ is equal to

$$
\begin{pmatrix}
-\frac{16}{t} & \frac{4}{t} & \cdots & \cdots & \frac{4}{t} & \frac{4}{t} & \cdots & \cdots & \frac{4}{t} & -\frac{4}{t} & \cdots & \cdots & -\frac{4}{t} & -\frac{4}{t} & \cdots & \cdots & -\frac{4}{t} \\
\frac{4}{t} & -\frac{4}{t}-\frac{4}{\varphi_1} & -\frac{4}{t} & \cdots & -\frac{4}{t} & & & & & & & & & & & & \\
\vdots & -\frac{4}{t} & \ddots & \ddots & \vdots & & & & & & & & & & & & \\
\vdots & \vdots & \ddots & & -\frac{4}{t} & & & & & & & & & & & & \\
\frac{4}{t} & -\frac{4}{t} & \cdots & -\frac{4}{t} & -\frac{4}{t}-\frac{4}{\varphi_p} & & & & & & & & & & & & \\
\frac{4}{t} & & & & & -\frac{4}{t}-\frac{4}{\varphi_1} & -\frac{4}{t} & \cdots & -\frac{4}{t} & & & & & & & & \\
\vdots & & & & & -\frac{4}{t} & \ddots & \ddots & \vdots & & & & & & & & \\
\vdots & & & & & \vdots & \ddots & & -\frac{4}{t} & & & & & & & & \\
\frac{4}{t} & & & & & -\frac{4}{t} & \cdots & -\frac{4}{t} & -\frac{4}{t}-\frac{4}{\varphi_p} & & & & & & & & \\
-\frac{4}{t} & & & & & & & & & -\frac{4}{t}-\frac{4}{\varphi_1} & -\frac{4}{t} & \cdots & -\frac{4}{t} & & & & \\
\vdots & & & & & & & & & -\frac{4}{t} & \ddots & \ddots & \vdots & & & & \\
\vdots & & & & & & & & & \vdots & \ddots & & -\frac{4}{t} & & & & \\
-\frac{4}{t} & & & & & & & & & -\frac{4}{t} & \cdots & -\frac{4}{t} & -\frac{4}{t}-\frac{4}{\varphi_p} & & & & \\
-\frac{4}{t} & & & & & & & & & & & & & -\frac{4}{t}-\frac{4}{\varphi_1} & -\frac{4}{t} & \cdots & -\frac{4}{t} \\
\vdots & & & & & & & & & & & & & -\frac{4}{t} & \ddots & \ddots & \vdots \\
\vdots & & & & & & & & & & & & & \vdots & \ddots & & -\frac{4}{t} \\
-\frac{4}{t} & & & & & & & & & & & & & -\frac{4}{t} & \cdots & -\frac{4}{t} & -\frac{4}{t}-\frac{4}{\varphi_p}
\end{pmatrix}
$$

Data availability

No data was used for the research described in the article.

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